

Dominance-based Solutions for Strategic Form Games

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Abstract

We model a player's decision as a choice set based on an abstract concept of dominance, called a *dominance structure*, and define a choice-theoretic notion of equilibrium. We investigate various properties of dominance structures and provide a general existence result; we give sufficient conditions for uniqueness of "maximal" and "minimal" equilibria; and we explore the logical relationships among several well-known and some new dominance structures. Our results explain many regularities observed in the literature on rationalizability, in which specific dominance structures are used to characterize rationalizable strategy profiles under different common knowledge assumptions. Our uniqueness result for "minimal" equilibria extends Shapley's (1964) uniqueness result for the saddle of a two-player zero-sum game.

1 Introduction

The focus of game theory, and the source of its richness, is the strategic indeterminacy inherent in many social situations. A long tradition, beginning with the work of von Neumann and Morgenstern (1944) and Nash (1951), resolves this indeterminacy by assigning mixed strategies to players and finding equilibrium points, assignments that allow no players to improve their expected payoffs. The equilibrium point approach rests on several assumptions: preferences over uncertain outcomes are given by expected utility; these preferences are common knowledge; rationality of the players is common knowledge; and the mixed strategies of the players are common knowledge.¹ Dominance concepts arise in attempts to weaken these assumptions, especially the last. Bernheim (1984) and Pearce (1984) drop common knowledge of mixed strategies and consider a strategy playable for player i if and only if it is rationalizable — it is a best response to an assignment of mixed strategies to i 's opponents, each other player j 's assigned mixed strategy is a best response to an assignment of mixed strategies to j 's opponents, and so on. Bernheim also considers the more demanding criterion of point rationalizability, which requires that a strategy be a best response to an assignment of pure strategies to i 's opponents and that each assignment satisfies the same condition. Börgers (1993) characterizes the playable strategies assuming only common knowledge of the players' ordinal preferences and rationality for some compatible von Neumann-Morgenstern preferences.² Alternative forms of rationality and non-expected utility preferences have been considered as well.³

In the rationalizability literature outlined above, the playable strategy profiles of a game are typically characterized by a particular concept of dominance and the iterative elimination of dominated strategies. Rationalizability in Bernheim and Pearce, for example, is characterized using the following concept.

¹The players are *rational* if they choose their strategies optimally given their preferences over uncertain outcomes and conjectures of the mixed strategies of their opponents. See Aumann and Brandenburger (1995) for recent work on the epistemic foundations of Nash equilibrium.

²See also Fishburn (1978) for an equilibrium point approach to this problem.

³Perfect and cautious rationalizability are defined by Bernheim and Pearce in an attempt to refine away less credible rationalizable strategies. Brandenburger (1992) and Stahl (1995) explore the bounds on rational play when players evaluate lotteries using lexicographic probability systems, as in Blume, Brandenburger, and Dekel (1991).

Rational dominance: Strategy x_i is dominated over Y_1, Y_2, \dots, Y_n if it is a best response to no mixed strategy profile with support in these sets.

Note that the status of x_i depends on the sets Y_1, Y_2, \dots, Y_n of strategies possible for i 's opponents, and that there need not be a particular strategy that dominates it. Iterative elimination of strategies dominated in this sense gives us the sets of rationalizable strategies for the players. Bernheim's point rationalizable strategy profiles are associated with a weaker dominance concept.

Point rational dominance: Strategy x_i is dominated over Y_1, Y_2, \dots, Y_n if it is a best response to no pure strategy profile in $Y_1 \times Y_2 \times \dots \times Y_n$.

Iteratively deleting strategies dominated in this sense gives us the sets of point rationalizable strategies. After dropping the assumption that expected utility preferences are common knowledge, Börgers (1993) uses a concept of dominance stronger than Bernheim and Pearce's.

Börgers dominance: Strategy x_i is dominated over Y_1, Y_2, \dots, Y_n if, for every collection Z_1, Z_2, \dots, Z_n of subsets, there is some pure strategy that weakly dominates it over those sets.

Again, iterative elimination of strategies dominated in Börgers's sense gives us the possible strategy profiles under his weaker common knowledge assumptions.

These treatments exhibit other interesting formal similarities. First, the sets remaining after the iterative elimination of dominated strategies, call them $Y_1^*, Y_2^*, \dots, Y_n^*$, are invariant with respect to the order in which dominated strategies are eliminated. Second, no strategies in Y_i^* are dominated over the other players' sets (an internal stability property) and all strategies outside Y_i^* are dominated (an external stability property). That is, each player's choice set is a "solution," given the choice sets of the other players. Third, the collection $(Y_1^*, Y_2^*, \dots, Y_n^*)$ is the unique maximal collection exhibiting these stability properties: if each component of $(Y_1', Y_2', \dots, Y_n')$ exhibits internal and external stability, then $Y_i' \subseteq Y_i^*$ for all i . Last, though the mixed strategies of the players are not common knowledge, the sets $Y_1^*, Y_2^*, \dots, Y_n^*$ are. That is, each player knows the sets of strategies from which his opponent may choose. These features are not common to all

dominance concepts. The sets remaining after iterative deletion of weakly dominated strategies, for example, are internally stable but do not typically possess any of the other properties.

In this paper, we start with a general formulation of dominance, called a *dominance structure*, from which we can obtain various notions of rationality and dominance as special cases. From this abstract perspective, we are able to unify many of the results known of different concepts of dominance — in particular, we explain the above-mentioned regularities in the rationalizability literature — and we obtain new insights into the play of strategic form games. We model the strategic decisions of players as choice sets and look for equilibria in terms of sets — collections $(Y_1^*, Y_2^*, \dots, Y_n^*)$ such that each Y_i^* has an internal and external stability property, given the choice sets of the other players.⁴ Thus, it is implicitly assumed that, along with the dominance structure of interest, the choice sets of the players are common knowledge. This is so for the sets of rationalizable strategies, described above, but also for other, non-maximal equilibria. We investigate several properties of general dominance structures, which we denote by Q , and their implications for our equilibria, which we call Q -solutions. We provide a general existence result and explore the relationships among the solutions of various dominance structures, some known and some new. And we establish conditions under which the maximal Q -solution is unique and invariant to the order of elimination of dominated strategies.

We also provide conditions sufficient for uniqueness of minimal Q -solutions, extending Shapley’s (1964) analysis of the “saddle” and “weak saddle” of two-player zero-sum games. The saddle is defined in two steps: first, a *generalized saddle* is a pair (Y_1, Y_2) of choice sets such that, for every strategy not in Y_1 , there is some strategy that strictly dominates it over Y_2 , and likewise for player 2; second, a *saddle* is a minimal generalized saddle, i.e. a pair (Y_1, Y_2) such that if (Z_1, Z_2) is a generalized saddle with $Z_1 \subseteq Y_1$ and $Z_2 \subseteq Y_2$ then $(Z_1, Z_2) = (Y_1, Y_2)$. The *weak saddle* is defined similarly, but using “weak” domination in place of “strict.” Thus, a generalized saddle is just a pair of choice sets with the external stability property. Clearly, the choice sets in a saddle must also exhibit

⁴The classical theory of choice, as in Arrow (1959), Richter (1966), Sen (1971), and Suzumura (1976), considers agents with fixed preferences. Interpreting “dominance” as an expression of strategic preference, our approach can be viewed as an extension to situations where an agent’s preferences may depend on the choice sets of others.

internal stability — otherwise, some player’s choice set contains a strictly dominated strategy that could be eliminated, resulting in a strictly smaller generalized saddle — so a saddle is just a minimal Q -solution by our definition, where Q is strict dominance. Shapley’s weak saddles need not exhibit internal stability, however, and do not necessarily correspond to solutions in our framework. Clearly, saddles and weak saddles exist in finite games — the pair of choice sets consisting of each player’s entire strategy set is a generalized saddle, and finiteness implies the existence of a minimal generalized saddle (similarly for the weak saddle). Moreover, Shapley proves that the saddle is also unique in two-player zero-sum games.⁵ Our uniqueness results for minimal Q -solutions generalize Shapley’s uniqueness result for the saddle to a class of multi-player non-zero-sum games possessing a *safe equilibrium* — a mixed strategy equilibrium (p_1, p_2, \dots, p_n) such that, for all i , p_i is a best response to all mixed strategy equilibrium profiles. Two-player zero-sum games are examples of “equilibrium safe” games, as is any game with a dominant strategy equilibrium or a unique mixed strategy equilibrium. As a consequence of our general approach, we deduce uniqueness of the minimal Q -solution for other dominance structures as well, including Bernheim and Pearce’s and Börgers’s.

Equilibria in terms of sets have been considered elsewhere. What we call “ P -solutions” and “ R -solutions” in Section 4 are the fixed points of Bernheim’s (1984) best response operators, λ and Λ , respectively. Samuelson’s (1992) “consistent pairs” are solutions under the premise that weakly dominated strategies are eliminated, what we call “ W -solutions” in Section 3. Closely related is the “weakly admissible set,” defined by McKelvey and Ordeshook (1976) and applied to spatial models of politics. As far as we know, however, Shapley alone takes up the issue of uniqueness of minimal solutions.

In Section 2 we outline a choice-theoretic approach to finite strategic form games, in which the strategic decisions of players are modelled as sets of acceptable strategies, or “choice sets.” We initially take as given a binary relation representing a single player’s strategic preferences and consider criteria by which a choice set might be constructed. One possibility is to suppose that strategies are acceptable if and only if they are maximal, but we focus on a more flexible criterion proposed by Wilson (1971), called the

⁵The weak saddle is not generally unique, though Duggan and Le Breton (1996) establish uniqueness in the subclass of symmetric games with non-zero off-diagonal payoffs.

“solution property.” It requires that no strategy chosen by a player be preferred to another chosen one, and that for every rejected strategy there is a chosen one preferred to it. We then define the concept of a dominance structure, which explicitly allows the strategic preferences of the players to depend on the choice sets of their opponents, and, for a given dominance structure Q , we define the Q -solution.

Section 3 begins our analysis of Q -solutions with general propositions on existence of and the logical relationships between the solutions corresponding to different dominance structures. The section makes ongoing reference to three dominance structures not explicitly defined above.

Shapley dominance: Strategy x_i dominates strategy y_i over Y_1, Y_2, \dots, Y_n if, for all profiles of strategies selected from these sets, x_i gives a strictly higher payoff than y_i .

Weak Shapley dominance: Strategy x_i dominates strategy y_i over Y_1, Y_2, \dots, Y_n if, for all profiles of strategies selected from these sets, x_i gives at least as high a payoff as y_i , strictly higher for some profile.

Nash dominance: Strategy x_i dominates strategy y_i over Y_1, Y_2, \dots, Y_n if, for all profiles of strategies selected from these sets, x_i gives player i at least as high a payoff as y_i .

The first of these dominance structures is used by Shapley (1964) to define the saddle and is quite strong. Shapley solutions are, therefore, typically quite large. As we show, weak Shapley solutions are generally subsets of Shapley solutions, and Nash solutions give predictions tighter still. The Shapley and Nash dominance structures possess two properties central to our analysis: they are transitive (as is weak Shapley) and *monotonic*, in the sense that if x_i dominates y_i over Y_1, Y_2, \dots, Y_n then it dominates y_i over any smaller sets Z_1, Z_2, \dots, Z_n . These conditions are sufficient for existence of Q -solutions. Weak Shapley dominance violates monotonicity, and we show that weak Shapley solutions may fail to exist as a consequence. The dominance structure of Börgers, closely related to weak Shapley dominance, is monotonic and transitive. Therefore, Börgers solutions, which lie between Shapley solutions and weak Shapley solutions, generally exist.

All of these solutions are weaker than mixed strategy Nash equilibrium in one sense — every Shapley, Börgers, weak Shapley, and Nash solution includes the support of at least one equilibrium. Shapley and Börgers solutions are also weaker in the sense that every mixed strategy equilibrium is embraced by at least one Börgers and one Shapley solution. Not so for Nash solutions: we give an example in which the support of a mixed strategy equilibrium has empty intersection with every Nash solution. The reason for this is that Nash solutions do not allow players to choose redundant strategies — strategies that give a player the same payoff over his opponents’ choice sets — whereas mixed strategy equilibria do. This suggests a possible refinement of Nash equilibrium that eliminates redundant strategies: keep the equilibria with support in some Nash solution and reject the others.

In Section 4, we show that a large class of transitive and monotonic dominance structures have unique maximal Q -solutions. Furthermore, the maximal Q -solution consists of the strategy profiles remaining after the iterative elimination of Q -dominated strategies, and these sets are invariant with respect to the order of elimination. This class of dominance structures includes rational dominance, point rational dominance, Börgers dominance, and Shapley dominance. We define other transitive and monotonic dominance structures, “leximin” and “maximin,” and we verify the logical relationships among the solutions of this section and the preceding one.

In Section 5, we extend our previously defined dominance structures to allow randomization by the players. Shapley dominance is extended, for example, to “mixed Shapley” dominance.

Mixed Shapley dominance: Strategy x_i is dominated over Y_1, Y_2, \dots, Y_n if there exists a mixed strategy that, for all profiles of strategies selected from these sets, gives a strictly higher payoff than x_i .

We show that the mixed Shapley solutions can be characterized as follows: (Y_1, Y_2, \dots, Y_n) is a mixed Shapley solution if, for all i , every strategy outside Y_i is strictly dominated over Y_1, Y_2, \dots, Y_n by some mixed strategy with support in Y_i , and no strategy in Y_i is dominated by such a mixed strategy. Thus, mixed Shapley solutions parallel Shapley solutions quite closely. Our earlier propositions apply to mixed Shapley dominance,

establishing existence and uniqueness of the maximal mixed Shapley solution.⁶

In Section 6, we investigate the uniqueness of minimal Q -solutions, or Q -sets. It is simple to verify that uniqueness does not hold generally: any strict Nash equilibrium is a Shapley set, weak Shapley set, Nash set, etc. Nonetheless, we show that the mixed Shapley set, the Börgers set, and the Shapley set are unique whenever the rational set, defined using the dominance structure of Bernheim and Pearce, is unique. We obtain uniqueness of the latter in equilibrium safe games, which include two-player zero-sum games. Thus, we extend Shapley's (1964) uniqueness result for the saddle. We give conditions sufficient for equilibrium safety, and we show how known concepts of equivalence among strategic form games allow us to extend our uniqueness results even further.

In Section 7, we extend our definition of Q -solution and our analysis of existence to games with arbitrary (possibly infinite) sets of players and compact topological spaces of strategies for the players. We find that monotonic dominance structures satisfying a combination of transitivity and upper semi-continuity have solutions. In particular, Shapley and Nash solutions exist when players' payoff functions are continuous. Unlike existence results for Nash equilibrium, convexity of preferences is unneeded. Proofs of our propositions are provided in an appendix. Figure 1 (attached) depicts the logical relationships among the Q -solutions considered in this paper.

2 A Choice-theoretic Approach

We consider a non-cooperative strategic form game $\Gamma = (I, (X_i)_{i \in I}, (u_i)_{i \in I})$, where I is a non-empty set of players, denoted i or j ; each X_i is a non-empty set of strategies, denoted x_i, y_i , etc.; and u_i is a payoff function defined on $X \equiv \prod_{i \in I} X_i$, with elements denote x, y , etc. Subsets of X_i are denoted Y_i or Z_i , and, given a collection $(Y_i)_{i \in I}$, we write $Y = \prod_{i \in I} Y_i$. (In the sequel, Y and Z will only denote such products.) Given a player j and a collection $(Y_i)_{i \neq j}$, we write Y_{-j} for $\prod_{i \neq j} Y_i$ and x_{-j}, y_{-j} , etc., for typical elements of Y_{-j} .

Following the classical theory of choice, we model a player i 's decision as a choice set, a set Y_i of strategies acceptable to i according to some criterion. Whereas the equilibrium

⁶These profiles are the correlated-rationalizable strategy profiles, in the fashion of the rationalizable profiles but allowing correlation in the mixed strategies of a player's opponents.

point approach requires players' preferences over lotteries, we need a different kind of information in addition to the primitives of the game Γ : how player i formulates Y_i . We assume that, given the choice sets of the other players, i 's "strategic preferences" are given by a binary relation Q_i on X_i , left unspecified for now — it may or may not be irreflexive, transitive, etc. Player i 's strategic preferences should be expected to reflect the information in u_i and to depend on the choice sets of the other players, but we first consider how a fixed Q_i determines a choice set for i . After that, we define a notion of equilibrium when the dependence of Q_i on Y_{-i} is explicitly acknowledged.

The usual criterion for the construction of i 's choice set is to suppose that Y_i consists of the maximal strategies according to Q_i (i.e., x_i is acceptable if and only if there is no y_i such that $y_i Q_i x_i$ and not $x_i Q_i y_i$), but Wilson (1970) proposes alternative notions in which a particular choice set may be consistent with a given binary relation. The first we consider, the "solution property," is equivalent to maximality whenever Q_i is transitive and irreflexive but gives us more flexibility in other cases.

Definition 1

(i) A set Y_i has the **inner solution property** with respect to Q_i if, for all $x_i \in X_i$,

$$(x_i \in Y_i) \Rightarrow (\forall y_i \in Y_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

(ii) A set Y_i has the **outer solution property** with respect to Q_i if, for all $x_i \in X_i$,

$$(x_i \notin Y_i) \Rightarrow (\exists y_i \in Y_i \setminus \{x_i\})(y_i Q_i x_i).$$

(iii) A set Y_i has the **solution property** with respect to Q_i if it has the inner and outer solution properties with respect to Q_i .

Interpreting Q_i as "dominates," Y_i has the solution property so long as x_i is chosen if and only if no other chosen strategy dominates it. Clearly, the solution property is equivalent to the following: for all $x_i \in X_i$,

$$(x_i \in Y_i) \Leftrightarrow (\forall y_i \in Y_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

Also, note that X trivially has the outer solution property. We call Y_i an *inner Q_i -solution* (resp. *outer Q_i -solution*, *Q_i -solution*) if it has the inner solution (resp. outer

solution, solution) property with respect to Q_i . A second notion of consistency defined by Wilson, the “core property,” coincides with the usual notion of maximality whenever Q_i is asymmetric.

Definition 2

(i) A set Y_i has the **inner core property** with respect to Q_i if, for all $x_i \in X_i$,

$$(x_i \in Y_i) \Rightarrow (\forall y_i \in X_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

(ii) A set Y_i has the **outer core property** with respect to Q_i if, for all $x_i \in X_i$,

$$(x_i \notin Y_i) \Rightarrow (\exists y_i \in X_i \setminus \{x_i\})(y_i Q_i x_i).$$

(iii) A set Y_i has the **core property** with respect to Q_i if it has the inner and outer core properties with respect to Q_i .

The core property is equivalent to the following: for all $x_i \in X_i$,

$$(x_i \in Y_i) \Leftrightarrow (\forall y_i \in X_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

Note that \emptyset trivially has the inner core property. We call Y_i an *inner Q_i -core* (resp. *outer Q_i -core*, *Q_i -core*) if it has the inner core (resp. outer core, core) property with respect to Q_i . There may generally be several subsets of X_i (or none) possessing the solution property with respect to Q_i , while there will be exactly one (possibly the empty set) satisfying the core property. In certain cases, however, the two properties are equivalent.

Proposition 1 *Assume Q_i is irreflexive and transitive. A set $Y_i \subseteq X_i$ has the solution property with respect to Q_i if and only if it has the core property with respect to Q_i .*

Of course, the status of i 's strategies will generally depend on the choices of other players, so we write $Q_i(Y_{-i})$ to make this dependence explicit. Formally, we view Q_i as a mapping from non-empty product sets Y_{-i} to binary relations on $\mathfrak{R}^{Y_{-i}}$, the set of real-valued functions on Y_{-i} . This induces a binary relation on X_i as follows:

$$y_i Q_i(Y_{-i}) x_i \Leftrightarrow u_i(y_i, \cdot) Q_i(Y_{-i}) u_i(x_i, \cdot).$$

We write Q for the mapping $\bigcup_{i \in I} Q_i$ defined on I , and we refer to Q as a *dominance structure*. A more direct definition would simply take $Q_i(Y_{-i})$ as a binary relation on X_i , but ours will facilitate the analysis of Section 5, where we consider extensions of dominance structures to allow players to use mixed strategies. In fact, we will often define dominance structures as binary relations over X_i , leaving a relation on $\mathfrak{R}^{Y_{-i}}$ to be inferred from our definitions.⁷ We now formalize our main solution concept for finite strategic form games.

Definition 3

- (i) A set Y is an **outer Q -solution** if, for each i , Y_i has the outer solution property with respect to $Q_i(Y_{-i})$.
- (ii) A set Y is an **inner Q -solution** if, for each i , Y_i has the inner solution property with respect to $Q_i(Y_{-i})$.
- (iii) A set Y is a **Q -solution** if it is both an outer Q -solution and an inner Q -solution.

At a Q -solution, players are best-responding in the sense that each i 's choice set consists of the acceptable strategies, given the choice sets Y_{-i} of the other players and strategic preferences $Q_i(Y_{-i})$. The flexibility of this definition allows us to consider various notions of “acceptability,” and their corresponding solutions, as we substitute one dominance structure for another. In later sections, we will use the term *Q -set* for a Q -solution Y that is minimal with respect to set-inclusion: if Z is a Q -solution and $Z \subseteq Y$ then $Z = Y$. In Section 7 we extend the definition of Q -solution to infinite games. We also have occasion to consider equilibria defined in terms of the core properties.

Definition 4

- (i) A set Y is an **outer Q -core** if, for each i , Y_i has the outer core property with respect to $Q_i(Y_{-i})$.
- (ii) A set Y is an **inner Q -core** if, for each i , Y_i has the inner core property with respect to $Q_i(Y_{-i})$.
- (iii) A set Y is a **Q -core** if it is both an outer Q -core and an inner Q -core.

⁷This imprecision will be corrected in future drafts.

Proposition 1 gives conditions under which Q -solutions are equivalent to Q -cores. The following proposition gives a useful characterization of Q -solutions. We define a dominance structure Q to be *transitive* (resp. *irreflexive*) if, for all i and all $Y_{-i} \subseteq X_{-i}$, $Q_i(Y_{-i})$ is transitive (resp. irreflexive).

Proposition 2 *If Y is a Q -solution then, for all i , Y_i is a minimal subset of X_i possessing the outer solution property with respect to $Q_i(Y_{-i})$. If Q is transitive, the converse is also true.*

This offers an interpretation of Q -solutions: given Y_{-i} , we can think of i as choosing the smallest possible subset of strategies having the outer solution property with respect to $Q_i(Y_{-i})$. In effect, i narrows down his choice set as far as possible, given the choice sets of the other players. In the next section we give examples of some dominance structures and begin our investigation of Q -solutions.

3 Q -Solutions

In this section, we prove general results for Q -solutions and apply them to several dominance structures. We focus at first on three: Shapley dominance, weak Shapley dominance, and Nash dominance. The first two of these extend notions defined by Shapley (1964) for two-player zero-sum games. The third extends the idea of pure strategy Nash equilibrium to settings where pure strategy equilibria may not exist. We find that Shapley and Nash solutions exist in general, but that weak Shapley solutions may not exist in some games. We then define a closely related dominance structure, due to Börgers (1993), for which solutions do generally exist. We investigate the interrelationships of these solutions and their relation to mixed strategy Nash equilibrium. The solutions, as expected, are typically weak, but Nash solutions are not unambiguously weaker than mixed strategy Nash equilibrium: we give an example in which there is a mixed strategy Nash equilibrium that is disjoint from every Nash solution. A byproduct of our results is a possible refinement of Nash equilibrium: retain only those mixed strategy equilibria with support in some Nash solution. (There is always at least one.) This refinement effectively eliminates “redundant” strategies from the support of mixed strategy equilibria. The results of this section lay the groundwork for the analysis of a wide variety of other dominance structures in later sections.

3.1 Shapley, Weak Shapley, and Nash Solutions

Three dominance structures important to our analysis are defined next. The first two, Shapley and weak Shapley dominance, are quite demanding — a dominated strategy *must*, in a sense, be rejected — while the third, Nash dominance, is only suggestive — a dominated strategy *can* be rejected without harm. In effect, Nash dominance eliminates redundancies from a player’s choice set.

(1) **Nash** Define $x_i N_i(Y_{-i}) y_i$ if and only if

$$(\forall x_{-i} \in Y_{-i})(u_i(x) \geq u_i(y_i, x_{-i})).$$

(2) **Weak Shapley** Define $x_i W_i(Y_{-i}) y_i$ if and only if

$$x_i N_i(Y_{-i}) y_i \text{ and } \neg y_i N_i(Y_{-i}) x_i.$$

(3) **Shapley** Define $x_i S_i(Y_{-i}) y_i$ if and only if

$$(\forall x_{-i} \in Y_{-i})(u_i(x) > u_i(y_i, x_{-i})).$$

Note the connection between N -solutions and pure strategy Nash equilibria: $\{x\}$ is a N -solution if and only if x is a pure strategy Nash equilibrium. (Of course, not all N -solutions are singletons.) Also note that $\{x\}$ is a S -solution (or W -solution) if and only if x is a strict Nash equilibrium. Clearly, the three structures coincide for generic games (where $u_i(x) \neq u_i(y)$ for all i , x , and y). All three dominance structures are transitive, and W and S are irreflexive as well. Thus, Proposition 1 implies that the S -solutions and S -cores coincide, and similarly for W -solutions and W -cores. The solution property offers no greater flexibility than the core property for these dominance structures, but it does for Nash dominance. See Example 2 in Subsection 3.3 for illustrations of the solutions of these different dominance structures.

Next, we define a condition central to our analysis of existence in this section and uniqueness of maximal and minimal Q -solutions in the sequel. Along with transitivity, it is one of the key conditions of the paper. The dominance structure Q is *monotonic* if, for all i , all $Y_{-i}, Z_{-i} \subseteq X_{-i}$, and all $x_i, y_i \in X_i$,

$$(Y_{-i} \subseteq Z_{-i} \wedge x_i Q_i(Z_{-i}) y_i) \Rightarrow (x_i Q_i(Y_{-i}) y_i).$$

In words, monotonicity says this: if x_i dominates y_i over a set Y_{-i} then it dominates y_i over any smaller set. Clearly, N and S are monotonic, though W is not. The next two dominance structures demonstrate the logical independence of transitivity and monotonicity. More transitive and monotonic dominance structures are defined in Sections 4 and 5.

(4) Define $x_i Q_i^m(Y_{-i}) y_i$ if and only if

$$\min_{z_{-i} \in Y_{-i}} u_i(x_i, z_{-i}) > \min_{z_{-i} \in Y_{-i}} u_i(y_i, z_{-i}).$$

(5) For an integer k , define $x_i Q_i^k(Y_{-i}) y_i$ if and only if

$$|\{z_{-i} \in Y_{-i} | u_i(y_i, z_{-i}) \geq u_i(x_i, z_{-i})\}| \leq k.$$

It can be seen that Q^m is transitive and irreflexive, while Q^k may not be; Q^k is monotonic, but Q^m may not be.

The next proposition reduces the existence problem for transitive, monotonic dominance structures to that of finding a minimal outer Q -solution. It is used throughout the paper to establish the logical relationships among various Q -solutions and known equilibrium concepts for finite games.

Proposition 3 *Let Q be transitive and monotonic. Consider an outer Q -solution Z and an inner Q -core Z' . If Y is minimal among the outer Q -solutions satisfying $Z' \subseteq Y \subseteq Z$, then Y is a Q -solution.*

Since X is trivially an outer Q -core and \emptyset is trivially an inner Q -core, an immediate implication of Proposition 3 is that, for transitive and monotonic dominance structures, Q -solutions exist. Specifically, S -solutions and N -solutions exist. This implies that W -solutions exist generically, but next is a counter-example to a general result for W -solutions. It is quite similar to Samuelson's (1992) Example 8, where he illustrates the possible inconsistency of common knowledge of "admissibility" (a generalization of our W -solutions). After the example, we will see how existence can be obtained for a dominance structure closely related to W .

Example 1 Let $|I| = 2$ and $X_1 = X_2 = \{a, b\}$, with payoffs specified below.

$$a \quad b$$

a	(2,1)	(1,2)
b	(1,2)	(1,1)

Suppose there is a W -solution $Y = Y_1 \times Y_2$. If $Y_2 = \{a, b\}$ then $Y_1 = \{a\}$, but then $Y_2 = \{b\}$. If $Y_2 = \{a\}$ then $Y_1 = \{a\}$, but then $Y_2 = \{b\}$. Finally, if $Y_2 = \{b\}$ then $Y_1 = \{a, b\}$, and then $Y_2 = \{a, b\}$.

The W -solution sometimes fails to exist because W is not generally monotonic. Börgers (1993) defines a closely related notion of dominance in his investigation of rationality in games when players know only the ordinal preferences of their opponents. Unlike W , this dominance structure is monotonic.

(6) **Börgers** Define $x_i B_i(Y_{-i}) y_i$ if and only if, for each $Z_{-i} \subseteq Y_{-i}$, there exists $z_i \in X_i$ such that $z_i W_i(Z_{-i}) y_i$.

Note that $x_i B_i(Y_{-i}) y_i$ offers grounds that y_i be rejected, but not necessarily because x_i should be chosen instead — this relation indicates not an advantage of x_i , but only a deficiency of y_i . Nonetheless, B is a well-behaved dominance structure by our criteria: it is transitive and monotonic, so Proposition 3 establishes existence of B -solutions. The definition of B suggests a general method for transforming a non-monotonic dominance structure Q into a monotonic one, the *monotonic kernel* of Q , denoted Q^\bullet and defined as follows: $x_i Q_i^\bullet(Y_{-i}) y_i$ if and only if, for all $Z_{-i} \subseteq Y_{-i}$, there is a $z_i \in X_i$ such that $z_i Q_i(Z_{-i}) y_i$. Clearly, $B = W^\bullet$.

3.2 Logical Relationships

The next example illustrates the solutions for the dominance structures of this section. It suggests a particular nesting of the solutions that is confirmed for the general case in the proposition following.

Example 2 Let $I = 2$, $X_1 = \{x_1, x_2, x_3\}$, and $X_2 = \{y_1, y_2, y_3, y_4, y_5\}$, with payoffs

specified below.

	y_1	y_2	y_3	y_4	y_5
x_1	(1,-1)	(-1,1)	(0,0)	(1,-1)	(0,-1)
x_2	(-1,1)	(1,-1)	(0,0)	(0,0)	(0,0)
x_3	(-1,1)	(1,-1)	(0,0)	(0,1)	(0,0)

Here $\{x_1, x_2\} \times \{y_1, y_2\}$ is not a N -solution, a W -solution, a B -solution, or a S -solution; $\{x_1, x_2\} \times \{y_1, y_2, y_3\}$ is a N -solution but not a W -solution, a B -solution, or a S -solution; $\{x_1, x_2, x_3\} \times \{y_1, y_2, y_3\}$ is a W -solution but not a solution for the other dominance structures; $\{x_1, x_2, x_3\} \times \{y_1, y_2, y_3, y_4\}$ is a B -solution but not a solution for the other dominance structures; and $X_1 \times X_2$ is a S -solution but not a solution for the other dominance structures.

In Example 2, the solutions for our dominance structures are unique and the relationships among them are unambiguous. However, in general the solutions of one dominance structure, Q , may be weaker than the solutions of another, Q' , in two ways: first, every Q -solution may be included in some Q' -solution, and second, every Q' -solution may include some Q -solution. If both statements are true, we say the Q -solutions are weaker than the Q' -solutions in the *full sense*. Proposition 3 yields the following relationships among the solutions of S , B , W , and N .

Theorem 1

- (i) If Y is a S -solution then there exists a B -solution $Z \subseteq Y$.
- (ii) If Y is a B -solution then there exists a N -solution $Z \subseteq Y$.
- (iii) If Z is a N -solution then there exists a B -solution $Y \supseteq Z$.
- (iv) If Z is a B -solution then there exists a S -solution $Y \supseteq Z$.

(v) If Y is a W -solution then there exist a N -solution Z , a B -solution Z' , and a S -solution Z'' such that $Z \subseteq Y \subseteq Z' \subseteq Z''$.

Proof: (i) Every S -solution is an outer B -solution, so Proposition 3 applies. (ii) Every B -solution is an outer N -solution, so Proposition 3 applies. (iii) Every N -solution is an inner B -core, Proposition 3 applies. (iv) Every B -solution is an inner S -core. (v) Every W -solution is an outer N -solution and an inner B -core. ■

Thus, Theorem 1 establishes that the relationships among the S -, B -, W -, and S -solutions suggested by Example 2 hold in the full sense for general games. See Figure 1 (attached). In the example, the set $\{x_1, x_2\} \times \{y_1, y_2\}$ is the support of the mixed strategy equilibrium in which row player plays x_1 and x_2 both with probability $1/2$ and column plays y_1 and y_2 with probability $1/2$. In the next subsection we will see that this nesting of mixed strategy equilibria within Q -solutions holds generally for S and B , and in one sense for N .

3.3 Mixed Strategy Nash Equilibrium

Given a strategic form game Γ , the *mixed extension* of Γ is the strategic form game $\tilde{\Gamma} = (I, (\tilde{X}_i)_{i \in I}, (\tilde{u}_i)_{i \in I})$, where \tilde{X}_i is the set of probability distributions (mixed strategies), denoted p_i or q_i , over X_i ; the set $\tilde{X} = \prod_{i \in I} \tilde{X}_i$ consists of strategy profiles, denoted p or q , in the mixed extension; and \tilde{u}_i is the real-valued function on \tilde{X} defined by

$$\tilde{u}_i(p) = \sum_{x \in X} \prod_{j \in N} p_j(x_j) u_i(x).$$

A *mixed strategy Nash equilibrium* of Γ is a Nash equilibrium of the mixed extension. That is, it is a profile p of mixed strategies such that, for all i and all $q_i \in \tilde{X}_i$, $\tilde{u}_i(p) \geq \tilde{u}_i(q_i, p_{-i})$. We will sometimes write $x_i \in \tilde{X}_i$ for the probability distribution with probability one on x_i . Fixing a mixed strategy profile p , we next define a dominance structure that allows us to characterize the support of the mixed strategy Nash equilibria of Γ in our framework.

(7) Define $x_i Q_i^p(Y_{-i}) y_i$ if and only if

$$\sum_{x_{-i} \in X_{-i}} \prod_{j \neq i} p_j(x_j | Y_j) u_i(x) > \sum_{x_{-i} \in X_{-i}} \prod_{j \neq i} p_j(x_j | Y_j) u_i(y_i, x_{-i}),$$

where $p_j(x_j|Y_j)$ is the probability j chooses x_j conditional on Y_j ; if $p_j(Y_j) = 0$, set $p_j(x_j|Y_j) = 0$.

This dominance structure is irreflexive and transitive but not monotonic, so Proposition 2 does not yield a Q^p -solution for arbitrary p . For $p_i \in \tilde{X}_i$, let $\sigma_i(p_i) = \{x_i \in X_i | p_i(x_i) > 0\}$ denote the support of p_i , and for $p \in \tilde{X}$, let $\sigma(p) = \prod_{i \in I} \sigma_i(p_i)$. As usual, $\sigma_{-i}(p_{-i}) = \prod_{j \neq i} \sigma_j(p_j)$.

Theorem 2 *The profile p is a mixed strategy Nash equilibrium if and only if $\sigma(p)$ is an inner Q^p -core.*

Proof: If p is a mixed strategy equilibrium, $x_i \in \sigma_i(p_i)$, and $y_i Q_i^p(Y_{-i})x_i$, then $\tilde{u}_i(y_i, p_{-i}) > \tilde{u}_i(x_i, p_{-i}) = \tilde{u}_i(p)$, a contradiction. Therefore, $\sigma(p)$ is an inner Q^p -core. Suppose $\sigma(p)$ is an inner Q^p -core but p is not an equilibrium, so there is some i and $q_i \in \tilde{X}_i$ such that $\tilde{u}_i(q_i, p_{-i}) > \tilde{u}_i(p)$. But, letting $x_i \in \sigma_i(p_i)$, there must then be some $y_i \in \sigma_i(q_i)$ such that $\tilde{u}_i(y_i, p_{-i}) > \tilde{u}_i(x_i, p_{-i})$, a contradiction. ■

The following theorem immediately implies that N -, B -, W -, and S -solutions are weaker than mixed strategy equilibrium in the sense that every such solution includes the support of some mixed strategy Nash equilibrium. (Note that all of these solutions are outer N -solutions.) Moreover, B -solutions and S -solutions are weaker in the full sense.

Theorem 3

(i) *If Y is an outer N -solution then there is a mixed strategy Nash equilibrium p such that $\sigma(p) \subseteq Y$.*

(ii) *If p is a mixed strategy Nash equilibrium then there is a B -solution $Y \supseteq \sigma(p)$.*

Proof: (i) Consider the restricted game, where each player's strategy set is Y_i and payoffs are given by u_i restricted to Y , and let p be a mixed strategy Nash equilibrium of this game. If it is not an equilibrium of the unrestricted game, there is some i and some $y_i \in X_i \setminus Y_i$ such that $\tilde{u}_i(y_i, p_{-i}) > \tilde{u}_i(p)$. Since Y is an outer N -solution, there exists some $x_i \in Y_i$ such that $x_i N_i(Y_{-i})y_i$, but then $\tilde{u}_i(x_i, p_{-i}) > \tilde{u}_i(p)$, and $\{p_i\}$ is not an equilibrium of the restricted game, a contradiction. (ii) Note that $\sigma(p)$ is an inner

B -core, for otherwise we have $x_i B_i(Y_{-i})y_i$, which implies $z_i W_i(\sigma_{-i}(p_{-i}))y_i$ for some $z_i \in X_i$. Thus, Proposition 3 applies. ■

The next example shows it is not generally the case that the support of every mixed strategy equilibrium is included in a N -solution. In fact, in the example there is a mixed strategy equilibrium with support disjoint from every N -solution. Thus, we can use N -solutions to refine mixed strategy Nash equilibria, effectively eliminating those in which redundant strategies are used: every game has at least one N -solution and every N -solution includes the support of at least one Nash equilibrium, so we can retain these equilibria and discard the others.

Example 3 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b, c, d\}$, with payoffs given below.

	a	b	c	d
a	(1,0)	(1,10)	(1,11)	(1,-1)
b	(1,10)	(1,0)	(1,-1)	(1,11)

Here, the mixed strategies that pick a and b each with probability $1/2$ constitute a mixed strategy Nash equilibrium with minimal support, $\{a, b\} \times \{a, b\}$. However, there is no N -solution intersecting $\{a, b\} \times \{a, b\}$: the only N -solutions are $\{a\} \times \{c\}$ and $\{b\} \times \{d\}$.

4 Uniqueness of Maximal Q -Solutions

In this section, we use Proposition 3 to establish conditions under which there exists a unique Q -solution Y maximal with respect to set-inclusion: if Z is a Q -solution then $Z \subseteq Y$. We then characterize the unique maximal Q -solution as the result of iteratively deleting Q -dominated strategies, and we show that, under our conditions on dominance structures, the order of deletion is irrelevant. Our general propositions immediately apply to Shapley dominance — the maximal S -solution is the set of strategy profiles remaining after iteratively deleting strictly dominated strategies — and they apply to Börgers dominance. We define two new dominance structures that yield the “point rationalizable sets” and “rationalizable sets” as their solutions, and our propositions then imply uniqueness of the maximal point rationalizable set and maximal rationalizable set, as in Bernheim (1984) and Pearce (1984). Finally, we define some other dominance structures that formalize different notions of rationality and suggest the broad applicability of our propositions.

The importance of uniqueness of maximal Q -solutions stems from common knowledge considerations: a unique maximal Q -solution describes the possible outcomes of a game when it is common knowledge among the players that each player i 's choice

set has the solution property with respect to $Q_i(Y_{-i})$, for some Y_{-i} . Thus, the assumption that choice sets themselves are common knowledge, implicit in the definition of Q -solution, does not prevent us from analyzing behavior under the weaker common knowledge assumption.

4.1 Uniqueness Results

The first proposition of this section, a consequence of Proposition 3, establishes uniqueness of maximal Q -solutions for a class of monotonic dominance structures. We say Q is *hard* if every Q -solution is an inner Q -core. Following Propositions 4-6, we give a sufficient condition, satisfied by S , B , W , and most of the dominance structures of the paper, for Q to be hard.

Proposition 4 *If Q is transitive, monotonic, and hard then the maximal Q -solution is unique.*

The next proposition characterizes the maximal Q -solution as the result of iteratively deleting Q -dominated strategy profiles. For each i and $Y_{-i} \subseteq X_{-i}$, let

$$U_i^{Q,1}(Y_{-i}) = \{x_i \in X_i \mid \neg \exists y_i \in X_i \ y_i Q_i(Y_{-i}) x_i\}.$$

Let $U^{Q,1}(Y) = \prod_{i \in I} U_i^{Q,1}(Y_{-i})$; let $U_i^{Q,k}(Y) = U_i^{Q,1}(U_{-i}^{Q,k-1}(Y))$; and let $U^{Q,k} = \prod_{i \in I} U_i^{Q,k}(Y)$. The sequence $\{U^{Q,k}(Y)\}$ is decreasing, and, because X is finite, there exists a K such that $U^{Q,K}(Y) = U^{Q,K+1}(Y) \equiv U^Q(Y)$.

Proposition 5 *If Q is transitive, monotonic, and hard then $U^Q(X)$ is the maximal Q -solution.*

We can also prove that, under the above conditions, the order of elimination does not matter.

Proposition 6 *Let Q be a transitive, monotonic, hard dominance structure. Let Y^1, Y^2, \dots, Y^K be a sequence of product sets decreasing to an inner Q -core Y^K such that $Y^1 = X$, and for all $k \geq 2$ and all i , Y^k is an outer $Q_i(Y_{-i}^{k-1})$ -core. Then $Y^K = U^Q(X)$.*

We call Q *weakly irreflexive* if, for all i , all $x_i, y_i \in X_i$, and all $Y_{-i} \subseteq X_{-i}$, $x_i Q_i(Y_{-i}) x_i$ implies $y_i Q_i(Y_{-i}) x_i$. In other words, thinking of $Q_i(Y_{-i})$ as a ranking, irreflexivities can occur only among bottom ranked strategies.

Proposition 7 *If Q is weakly irreflexive, transitive, and, for all i and all $Y_{-i} \subseteq X_{-i}$, the $Q_i(Y_{-i})$ -core is non-empty, then Q is hard.*

Thus, Propositions 4 and 5 apply to S -solutions, yielding a characterization of the unique maximal S -solution in terms of the iterative elimination of strictly dominated strategies. Proposition 6 gives us the well-known invariance of this set with respect to the order of elimination. Our characterization admits only dominating *pure* strategies — if a strategy is strictly dominated by a mixed strategy but not a pure one, it is not eliminated. Accounting for mixed strategies rules out some strategy profiles that N -solutions (and therefore W -, B -, or S -solutions) may not.

Example 4 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{a, b, c\}$, with payoffs as below.

	a	b	c
a	(1,3)	(2,0)	(1,1)
b	(2,0)	(1,3)	(1,1)

Here, $\{a, b\} \times \{a, b, c\}$ is the unique N -solution, but c is strictly dominated for player 2 by the mixed strategy that chooses a and b with equal probability.

Börger and Samuelson’s (1992) Example 4 demonstrates that there may be multiple maximal W -solutions, but our propositions do apply to B -solutions. With the results of Börger (1993), we conclude that the maximal B -solution consists of just the strategy profiles possible when it is common knowledge among the players that each knows the others’ ordinal preferences and each maximizes expected utility for compatible some von Neumann-Morgenstern utility function.

4.2 Point Rationalizability, Maximin, Leximin

Propositions 4-6 also have consequences for Bernheim’s (1984) notion of point rationalizability. Given Y_{-i} , let $BP_i(Y_{-i})$ denote the set of i ’s pure strategy best responses

to pure strategy profiles $y_{-i} \in Y_{-i}$. That is, $x_i \in BP_i(Y_{-i})$ if and only if there exists $y_{-i} \in Y_{-i}$ such that

$$u_i(x_i, y_{-i}) = \max_{y_i \in X_i} u_i(y).$$

We say Y is a *point rationalizable set* if, for all i , $Y_i = BP_i(Y_{-i})$. A strategy x_i is *point rationalizable* if $x_i \in Y_i$ for some point rationalizable set Y . Consider the following dominance structure.

(8) **Point Rational** Define $x_i P_i(Y_{-i}) y_i$ if and only if $y_i \notin BP_i(Y_{-i})$.

Like Börgers dominance, $x_i P_i(Y_{-i}) y_i$ offers grounds that y_i be rejected, but not necessarily because x_i should be chosen instead. Nonetheless, a P -solution consists of just the strategies that are satisfactory by the point rationality criterion, as the next theorem shows.

Theorem 4 *The set Y is a point rationalizable set if and only if it is a P -solution.*

Proof: Let Y be a point rationalizable set, and take any $x_i, y_i \in Y_i$. Clearly, $x_i P_i(Y_{-i}) y_i$ cannot hold. Now take any $x_i \notin Y_i$, so that $x_i \notin BP_i(Y_{-i})$. Let $y_i \in X_i$ be a best response to some $x_{-i} \in Y_{-i}$, so that $y_i \in Y_i$, and note that $y_i P_i(Y_{-i}) x_i$. Thus, Y is a P -solution. Now let Y be a P -solution, and take any $x_i \in BP_i(Y_{-i})$. If $x_i \notin Y_i$ then $y_i P_i(Y_{-i}) x_i$ for some $y_i \in Y_i$, an impossibility. And if $x_i \notin BP_i(Y_{-i})$ then, again, there is some $y_i \in BP_i(Y_{-i})$, and therefore $y_i P_i(Y_{-i}) x_i$. ■

The dominance structure P is transitive (trivially) and monotonic, so Proposition 3 yields existence of P -solutions. It is hard as well, so Propositions 4 and 5 characterize the unique maximal P -solution as the result of iteratively deleting strategies that are not best responses to pure strategy profiles of other players. We next define a dominance structure that, like P , is transitive (trivially) and monotonic, so Proposition 3 applies again. It is hard as well, so Propositions 4 and 5 characterize the unique maximal solution.

(9) **Maximin** Define $x_i M_i(Y_{-i}) y_i$ if, for all $Z_{-i} \subseteq Y_{-i}$, there exists $z_i \in X_i$ such that

$$\min_{y_{-i} \in Z_{-i}} u_i(z_i, y_{-i}) > \min_{y_{-i} \in Z_{-i}} u_i(y).$$

In words, $x_i M_i(Y_{-i}) y_i$ if, for any subset of Y_{-i} , the minimum payoff of y_i over that subset is less than the minimum payoff of some other strategy. Another interesting dominance structure is defined next. Given two real-valued functions v and v' defined on a subset $Z_{-i} \subseteq X_{-i}$, let $Z_{-i}(v, v') = \{y_{-i} \in Z_{-i} | v(y_{-i}) \neq v'(y_{-i})\}$. We write $v L_i(Z_{-i}) v'$ if $Z_{-i}(v, v') \neq \emptyset$ and $\min\{v(z_{-i}) | z_{-i} \in Z_{-i}(v, v')\} > \min\{v'(z_{-i}) | z_{-i} \in Z_{-i}(v, v')\}$.

(10) **Leximin** Define $x_i L_i(Y_{-i}) y_i$ if, for all $Z_{-i} \subseteq Y_{-i}$, there exists $z_i \in X_i$ such that

$$u_i(z_i, \cdot) L_i(Z_{-i}) u_i(y_i, \cdot).$$

This dominance structure is similar to M , but in case the minimum payoffs of z_i and y_i over Z_{-i} are tied, we compare their next lowest payoffs, or if those are tied, the next lowest, and so on. The solutions of L , as we will see, lie somewhere between those of P and M . Like M , it is transitive, monotonic, and hard. The definitions of P , M , and L suggest a family of transitive, monotonic, hard dominance structures. For example, we could define “Maximax” by replacing “min” with “max” in the definition of M . Or we could require that the maximum *and* minimum payoff of z_i over Z_{-i} are greater than the maximum and minimum payoff, respectively, of y_i over Z_{-i} . Or we could require that, for each Z_{-i} there exist z_i and z'_i such that the minimum payoff of z_i over Z_i is higher than the minimum of y_i , and the maximum of z'_i over Z_{-i} is higher than the maximum of y_i . Our propositions apply to these dominance structures as well.

4.3 Rationalizability and Cautious Rationalizability

Given Y_{-i} , let $BR_i(Y_{-i})$ denote the set of i 's pure strategy best responses to *mixed* strategy profiles $p_{-i} \in \tilde{Y}_{-i}$. That is, $x_i \in BR_i(Y_{-i})$ if and only if there exists $p_{-i} \in \tilde{Y}_{-i}$ such that

$$\tilde{u}_i(x_i, p_{-i}) = \max_{y_i \in X_i} \tilde{u}_i(y_i, p_{-i}).$$

We say Y is a *rationalizable set* if, for all i , $Y_i = BR_i(Y_{-i})$. A strategy x_i is *rationalizable* if $x_i \in Y_i$ for some rationalizable set Y . We characterize rationalizability using the following dominance structure.

(11) **Rational** Define $x_i R_i(Y_{-i}) y_i$ if and only if $y_i \notin BR_i(Y_{-i})$.

As with P , the dominance structure R is transitive (trivially) and monotonic, so Proposition 3 yields existence of R -solutions. It is hard, so Propositions 4 and 5 characterize the unique maximal R -solution as the result of iteratively deleting strategies that are not best responses to mixed strategy profiles of other players. The next theorem parallels Theorem 4, characterizing the R -solutions as the rationalizable sets.

Theorem 5 *The set Y is a rationalizable set if and only if it is a R -solution.*

Proof: Let Y be a rationalizable set, and take any $x_i, y_i \in Y_i$. Clearly, $x_i R_i(Y_{-i}) y_i$ cannot hold. Now take any $x_i \notin Y_i$, so that $x_i \notin BR_i(Y_{-i})$. Let $y_i \in X_i$ be a best response to some $p_{-i} \in \tilde{Y}_{-i}$, so that $y_i \in Y_i$, and note that $y_i R_i(Y_{-i}) x_i$. Thus, Y is a R -solution. Now let Y be a R -solution, and take any $x_i \in BR_i(Y_{-i})$. If $x_i \notin Y_i$ then $y_i R_i(Y_{-i}) x_i$ for some $y_i \in Y_i$, an impossibility. And if $x_i \notin BR_i(Y_{-i})$ then, again, there is some $y_i \in BR_i(Y_{-i})$, and therefore $y_i R_i(Y_{-i}) x_i$. ■

Like S -solutions and B -solutions, R -solutions are weaker than mixed strategy Nash equilibrium in the full sense, as the next theorem states. This is not necessarily true of the dominance structures of the previous subsection. The proof uses Proposition 3 and is omitted.

Theorem 6

- (i) *If Y is an outer R -solution then there is a mixed strategy Nash equilibrium p such that $\sigma(p) \subseteq Y$.*
- (ii) *If p is a mixed strategy Nash equilibrium then there is a R -solution Y such that $Y \supseteq \sigma(p)$.*

Pearce (1984) defines a related type of rationalizability, called “cautious rationalizability,” that roughly combines rationalizability with iterative elimination of weakly dominated strategies. We can characterize it using the following dominance structure. Given $Y \subseteq X$, let \tilde{Y}° denote the mixed strategy profiles p with support equal to Y , i.e. $\sigma(p) = Y$.

- (12) **Cautious Rational** Define $x_i C_i(Y_{-i}) y_i$ if and only if $y_i \notin BR(\tilde{Y}_{-i}^\circ)$.

That is, $x_i C_i(Y_{-i}) y_i$ when y_i is a best response to no strategy profile completely mixed on Y_{-i} . Pearce's (1984) Lemma 4 and Myerson's (1991) Theorem 1.7 show that, when $|I| = 2$, $x_i C_i(Y_{-i}) y_i$ if and only if $z_i W_i(Y_{-i}) y_i$ for some z_i . Thus, C -solutions are equivalent to W -solutions in two-player games and might be expected to inherit some of the difficulties of W -solutions. Indeed, C is transitive but not monotonic, and Example 1 demonstrates that C -solutions need not exist. Pearce defines the *cautious rationalizable* strategy profiles as those remaining after iteratively deleting C -dominated strategy profiles from the rationalizable ones: they comprise the set $U^C(U^R(X))$. This set is generally an inner C -solution but not an outer C -solution. Using the monotonic kernel of C does not provide a novel dominance structure, for it can be checked that $C^\bullet = R$.

4.4 Logical Relationships

The next example illustrates how the iterative procedure of Proposition 5 isolates the unique maximal P -, R -, M -, and L -solutions, and it demonstrates the conceptual distinctness of these solutions, from each other and from our earlier dominance structures: the maximal solutions for the dominance structures defined in this section are not inner W -solutions (and therefore not inner N -solutions) and they are not outer N -solutions (and therefore not outer W -, B -, or S -solutions).

Example 5 Let $|I| = 2$, $X_1 = \{x_1, x_2, x_3, x_4, x_5\}$, and $X_2 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, with payoffs as below.

	y_1	y_2	y_3	y_4	y_5	y_6
x_1	(1,5)	(1,5)	(1,0)	(3,4)	(3,3)	(1,4)
x_2	(2,5)	(2,4)	(1,5)	(3,0)	(2,3)	(1,4)
x_3	(3,0)	(3,0)	(3,1)	(1,2)	(1,3)	(1,2)
x_4	(0,0)	(0,0)	(2,0)	(2,0)	(0,1)	(0,0)
x_5	(0,0)	(0,0)	(1,0)	(2,0)	(0,1)	(0,0)

The maximal P -solution is constructed as follows: in the first round x_4 , x_5 , and y_6 are eliminated, leaving $\{x_1, x_2, x_3\} \times \{y_1, y_2, y_3, y_4, y_5\}$.

The maximal R -solution is constructed similarly: in the first round x_4 and x_5 are eliminated, but y_6 is not. The latter strategy is a best response to the mixed strategy placing probability $1/3$ on x_1 , x_2 , and x_3 . Thus, the unique maximal R -solution is $\{x_1, x_2, x_3\} \times \{y_1, y_2, y_3, y_4, y_5, y_6\}$.

The unique maximal L -solution is constructed as follows: in the first round x_5 and y_6 are eliminated, leaving $\{x_1, x_2, x_3, x_4\} \times \{y_1, y_2, y_3, y_4, y_5\}$. Here, x_5 is easily dominated over all sets of strategies for column player except for $\{y_3, y_4\}$. It is leximin dominated over this set by x_1 (and x_2 and x_3): the minimum payoff of x_1 and x_5 over $\{y_3, y_4\}$ is 1, and the next lowest payoff of x_5 over $\{y_3, y_4\}$ is 2, while the next lowest payoff of x_1 over $\{y_3, y_4\}$ is 3. The strategy x_4 is not eliminated, because there does not exist a strategy that leximin dominates x_4 over $\{y_3, y_4\}$.

The unique maximal M -solution is just $\{x_1, x_2, x_3, x_4, x_5\} \times \{y_1, y_2, y_3, y_4, y_5\}$. Though y_6 can still be eliminated, x_5 cannot be.

Example 5 suggests a nesting of the solutions corresponding to the dominance structures of this section. Proposition 3 yields the following theorem, which confirms the alleged relationships for S -, M -, L -, and P -solutions. The proof is straightforward, mimicking the proof of Theorem 1, and is omitted.

Theorem 7

- (i) If Y is a S -solution then there exists a M -solution $Z \subseteq Y$.
- (ii) If Y is a M -solution then there exists a L -solution $Z \subseteq Y$.
- (iii) If Y is a L -solution then there exists a P -solution $Z \subseteq Y$.
- (iv) If Z is a P -solution then there exists a L -solution $Y \supseteq Z$.
- (v) If Z is a L -solution then there exists a M -solution $Y \supseteq Z$.
- (vi) If Z is a M -solution then there exists a S -solution $Y \supseteq Z$.

Thus, L -solutions are nested within M -solutions. The next theorem establishes that L -solutions are nested within B -solutions as well, though Example 2 shows that M -solutions are generally not: in the example, $\{x_1, x_2, x_3, x_4\} \times \{y_1, y_2, y_3, y_4, y_5, y_6\}$ is a B -solution. The proof is omitted.

Theorem 8

- (i) If Y is a B -solution then there exists a L -solution $Z \subseteq Y$.
- (ii) If Z is a L -solution then there exists a B -solution $Y \supseteq Z$.

We also obtain a nesting of the S -, B -, R -, and P -solutions. Example 5 shows that no nesting of R - with M - or L -solution holds in general. Again, the proof is omitted.

Theorem 9

- (i) If Y is a B -solution then there exists a R -solution $Z \subseteq Y$.
- (ii) If Y is a R -solution then there exists a P -solution $Z \subseteq Y$.
- (iii) If Z is a P -solution then there exists a R -solution $Y \supseteq Z$.
- (iv) If Z is a R -solution then there exists a B -solution $Y \supseteq Z$.

See Figure 1 (attached). Theorems 7 and 9 clearly demonstrate the value of Proposition 3 in comparing the solutions of different dominance structures. Applying the proposition in the by-now-familiar way, we also see that, if a C -solution exists, it is included in a R -solution. C -solutions do not necessarily include P -solutions, however, as the next example shows.

Example 6 Let $|I| = 2$, $X_1 = \{a, b\}$, and $X_2 = \{c, d\}$, with payoffs specified below.

	c	d
a	(2,0)	(1,0)
b	(1,0)	(1,0)

Here, $\{a\} \times \{c, d\}$ is a C -solution, but there is no P -solution included in it: in any P -solution Y , c and d must both belong to column player's choice set, but then, since b is a best response to d , a and b must belong to row player's. Therefore, $Y = \{a, b\} \times \{c, d\}$.

5 Mixed Q -solutions

Thus far we have examined solutions defined using notions of dominance between pure strategies. A strategy x_i is eliminated from a S -solution Y , for example, if there is another y_i that strictly dominates x_i over Y_{-i} . We now explore the consequences of allowing players to use mixed strategies in formulating their choice sets. Given a dominance structure Q for a finite game Γ , we define its extension to the mixed extension $\tilde{\Gamma}$, and we use this to define “mixed Q ”: this is a dominance structure for Γ that captures the essence of Q when players are allowed to use mixed strategies and have expected utility preferences. Mixed solutions are still defined in terms of choice sets of pure strategies — choice sets of the players are assumed to be common knowledge but, in contrast to the equilibrium point approach, the mixed strategies of the players need not be.

We establish the logical relationships among mixed Q -solutions and compare them

to their pure counterparts: not surprisingly, Q -solutions are weaker than mixed Q -solutions. Applying Propositions 4 and 5 to the mixed S -solutions (which coincide with the mixed B -solutions), we find that the maximal mixed S -solution is unique: it is the result of iteratively deleting strictly dominated strategies, where we now admit domination by mixed strategies. In Section 8 (to be added later), we explore the connection between the mixed Q -solutions of a game and the Q -solutions of its mixed extension.

Given a dominance structure Q for a strategic form game Γ , we define the *extension* of Q , denoted \tilde{Q} , to the mixed extension $\tilde{\Gamma}$ as follows:

$$\tilde{u}_i(p_i, \cdot) \tilde{Q}_i(Y_{-i}) \tilde{u}_i(q_i, \cdot) \Leftrightarrow \tilde{u}_i(p_i, \cdot) Q_i(Y_{-i}) \tilde{u}_i(q_i, \cdot).$$

Note that the relation $p_i \tilde{Q}_i(Y_{-i}) q_i$ only holds for collections Y_{-i} of pure strategy profiles. For sets of non-degenerate mixed strategy profiles, p_i and q_i are non-comparable according to this dominance structure. The extension \tilde{Q} inherits many of the properties of Q : irreflexivity, transitivity, monotonicity, etc. We now use the extension of Q to construct a related dominance structure for Γ .

(13) **Mixed Q** Define $x_i Q_i^*(Y_{-i}) y_i$ if and only if $p_i \tilde{Q}_i(Y_{-i}) y_i$ for some $p_i \in \tilde{X}_i$.

Applied to our earlier dominance structures, this gives us a variety of mixed dominance structures. Of special interest later in this section is S^* , the mixed Shapley dominance structure, which is easily seen to coincide with B^* , the mixed Börgers dominance structure: $S^* = B^*$. The next proposition establishes the important properties of mixed dominance structures.

Proposition 8 *Let Q be a dominance structure for Γ .*

- (i) Q^* is transitive and weakly irreflexive.
- (ii) If Q is monotonic then Q^* is monotonic.
- (iii) If, for all i and all $Y_{-i} \subseteq X_{-i}$, the $\tilde{Q}_i(Y_{-i})$ -core is non-empty, then Q^* is hard.

Thus, Q^* automatically satisfies several important properties and inherits others from Q . Since S^* is hard, Propositions 4 and 5 yield uniqueness of the maximal S^* -solution and a characterization in terms of iterative elimination of dominated strategies. The next proposition gives Q^* -solutions a more intuitive interpretation. We say Q is

stratified if, for all i , all $p_i, q_i \in \tilde{X}_i$, and all $Y_{-i} \subseteq X_{-i}$ such that $p_i \tilde{Q}_i(Y_{-i}) q_i$, there exists $\tilde{p}_i \in \tilde{X}_i$ such that $\tilde{p}_i \tilde{Q}_i(Y_{-i}) q_i$ and $\sigma_i(\tilde{p}_i)$ is an inner $Q_i^*(Y_{-i})$ -core.

Proposition 9 *Assume Q is stratified. The set Y is a Q^* -solution if and only if the following two conditions hold.*

- (i) *For all i and all $x_i \notin Y_i$, there exists $p_i \in \tilde{Y}_i$ such that $p_i \tilde{Q}_i(Y_{-i}) x_i$.*
- (ii) *For all i and all $x_i \in Y_i$, there is no $p_i \in \tilde{Y}_i$ such that $p_i \tilde{Q}_i(Y_{-i}) x_i$.*

The dominance structures S , B , W , and N are all stratified. This gives us, for example, the following characterization of S^* -solutions: Y is a S^* -solution if and only if, for all i and all Y_{-i} ,

- (1) $x_i \notin Y_i$ implies there exists $p_i \in \tilde{Y}_i$ such that, for all $y_{-i} \in Y_{-i}$, $\tilde{u}_i(p_i, y_{-i}) > \tilde{u}_i(x_i, y_{-i})$, and
- (2) $x_i \in Y_i$ implies there is no such $p_i \in \tilde{Y}_i$.

Viewed this way, Q^* -solutions look very much like Q -solutions, except that domination of strategies outside Y_i by mixed strategies with support in Y_i is permitted, and domination of strategies inside Y_i by mixed strategies with support in Y_i is excluded. Proposition 3 yields the following relationships between S^* -solutions and the solutions of the previous sections.

Theorem 10

- (i) *If Y is a B -solution then there is a S^* -solution $Z \subseteq Y$.*
- (ii) *If Y is a S^* -solution then there is a R -solution $Z \subseteq Y$.*
- (iii) *If Z is a R -solution then there is a S^* -solution $Y \supseteq Z$.*
- (iv) *If Z is a S^* -solution then there is a B -solution $Y \supseteq Z$.*

Proof: We prove only (i) and (iv). We first claim that every B -solution Y is an outer S^* -solution. Take i and $x_i \notin Y_i$, and index the product subsets of Y_{-i} as $Z_{-i}^1, \dots, Z_{-i}^K$. For each k , there exists $y_i^k \in Y_i$ such that $y_i^k W_i(Z_{-i}^k) x_i$. Defining $p_i \in \tilde{Y}_i$ by placing

probability $1/K$ on each y_i^k , we see that $p_i \tilde{Q}_i(Y_{-i})x_i$, as claimed. Then Proposition 3 applies, establishing (i). The same argument shows that every S^* -solution is an inner B -core, so Proposition 3 yields (iv). ■

The next theorem illustrates the close relationship between these dominance structures in two-player games. The proof is straightforward, using Pearce's (1984) Lemma 2 or Myerson's (1991) Theorem 1.6, and is omitted.

Theorem 11 *If $|I| = 2$ then $S^* = B^* = R$.*

Theorems 6 and 10 imply that S^* -solutions are weaker in the full sense than mixed strategy Nash equilibrium. The relationship between N -solutions is only partially determined by Proposition 3: every R -solution (and therefore every S^* -solution) includes a N -solution. The next example shows that a R -solution (and therefore a S^* -solution) may not have a N -solution embracing it.

Example 7 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c, d\}$, with payoffs given below.

	a	b	c	d
a	(0,0)	(1,-1)	(1,-1)	(-1,1)
b	(-1,1)	(0,0)	(1,-1)	(-1,1)
c	(-1,1)	(-1,1)	(0,0)	(1,-1)
d	(1,-1)	(1,-1)	(-1,1)	(0,0)

Here, $X_1 \times X_2$ is a R -solution, but it is not a N -solution, since $aN_1(X_2)b$. In fact, $aW_1(X_2)b$, and it is not a W -solution either.

6 Uniqueness of Minimal Q -solutions

Given a dominance structure Q , the tightest possible predictions in our framework are those of the minimal Q -solutions, or Q -sets. Arbitrary games may clearly have multiple S -, B -, S^* -, and R -sets: since every strict Nash equilibrium is a Q -set for each of these dominance structures, coordination games give us examples of non-uniqueness. Shapley (1964) proves, however, that every two-player zero-sum game has a unique S -set. In this section, we extend Shapley's result to a wider class of strategic form games, called "equilibrium safe," and to other dominance structures: we show that the S -, B -, S^* -, and R -set are unique in equilibrium safe games. This class of games includes multi-player games characterized, broadly speaking, by the absence of equilibrium coordination problems. We give several sufficient conditions for a game to be equilibrium safe. At the end of the section, we make use of well-known equivalence relations on games to extend our uniqueness results to an even wider class of games.

6.1 Uniqueness Results

Note that the R -sets in the example of coordination games are singletons, so that no two have non-empty intersection. The next proposition establishes that this is true in general.

Proposition 10 *Assume Q is weakly irreflexive, transitive, and monotonic. If Y and Z are distinct Q -sets, then $Y \cap Z = \emptyset$.*

Of course, S , S^* , and R are weakly irreflexive, so Proposition 10 applies. It turns out that we can restrict our investigation of uniqueness to R -sets, for uniqueness of the other solutions follows. We say Q is *heavier* than Q' if every Q -solution is an outer Q' -solution.

Proposition 11 *Let Q' be transitive and monotonic. If Q is heavier than Q' and the Q' -set is unique, then the Q -set is unique.*

It can be checked that S is heavier than B , which is heavier than S^* , which is heavier than R . The class of games to which our theorems apply is defined next.

Definition 5 *A strategic form game is **equilibrium safe** if there exists a mixed strategy Nash equilibrium p^* such that, for all mixed strategy equilibria p and all i , $\tilde{u}_i(p_i^*, p_{-i}) \geq \tilde{u}_i(p)$.*

We call an equilibrium as in Definition 5 *safe*. Thus, if player i anticipates that the other players will play some equilibrium p , i 's expected payoff is no worse playing a safe equilibrium strategy. The next subsection contains a variety of sufficient conditions for equilibrium safety, but three are immediately apparent: games with unique Nash equilibria, games with any dominant strategy equilibria, and two-player zero-sum games are equilibrium safe. The main result of this section is now quick work. Since every two-player zero-sum game is equilibrium safe, Theorem 12 generalizes Shapley's (1964) uniqueness result for the Shapley set.

Theorem 12 *If Γ is equilibrium safe then the R -set is unique, the S^* -set is unique, the B -set is unique, and the S -set is unique.*

Proof: We claim that if Y is an outer R -solution and p^* is a safe equilibrium, then $\sigma(p^*) \subseteq Y$. To see this, let $p \in \tilde{Y}$ be a mixed strategy Nash equilibrium of the restricted game $\Gamma' = (I, (Y_i)_{i \in I}, (u'_i)_{i \in I})$, where u'_i is the restriction of u_i to Y . That p is a mixed strategy Nash equilibrium of Γ follows since, by the outer solution property with respect to $R_i(Y_{-i})$, i 's best responses to p_{-i} are contained in \tilde{Y}_i . By the definition of equilibrium safety, p_i^* is a best response to p_{-i} , so $\sigma_i(p_i^*) \subseteq Y_i$. Now, if Y and Z are R -sets, and therefore outer R -solutions, our claim implies that $\sigma(p^*) \subseteq Y \cap Z \neq \emptyset$. Then Proposition 10 implies $Y = Z$. Uniqueness of the B -, S^* -, and S -sets follows from Proposition 11. ■

Equilibrium safety is not necessary for uniqueness of S -, B -, S^* -, or R -sets, as the next example shows.

Example 8 Let $|I| = 2$, $X_1 = X_2 = \{a, b\}$, with payoffs as below.

	a	b
a	(2,1)	(1,1)
b	(1,1)	(2,1)

Here, $\{a, b\} \times \{a, b\}$ is the unique R -solution, and therefore the unique S^* -, B -, and S -solution as well. However, this game is not equilibrium safe: no other strategy gives player 1 as high a payoff as a when player 2 picks a , but, when player 2 picks b , b gives player 1 a strictly higher payoff than a .

The next example shows that uniqueness does not hold among the S -, B -, S^* -, and R -solutions generally. Because the example is highly structured, there appear to be no reasonable conditions on games that would ensure uniqueness of non-minimal solutions.

Example 9 Let $|I| = 2$, $X_1 = X_2 = \{a, b, c, d\}$, with zero-sum payoffs as below.

	a	b	c	d
a	(0,0)	(1,-1)	(-1,1)	(-1,1)
b	(-1,1)	(0,0)	(-1,1)	(1,-1)
c	(1,-1)	(1,-1)	(0,0)	(1,-1)
d	(1,-1)	(-1,1)	(-1,1)	(0,0)

Here $X_1 \times X_2$ is a S -, B -, S^* -, and R -solution, but it is not minimal — the unique Q -set for all of these dominance structures is $\{c\} \times \{c\}$.

6.2 Equilibrium Safety

Our analysis of equilibrium safe games focuses on two different conditions, both sufficient for equilibrium safety.

Definition 6

- (i) A strategic form game is **equilibrium interchangeable** if, for all mixed strategy Nash equilibria p and p' and all i , (p_i, p'_{-i}) is an equilibrium.
- (ii) A strategic form game is **safe** if (a) there is a unique equilibrium payoff vector $(\bar{u}_i)_{i \in I}$, and (b) there is a mixed strategy Nash equilibrium \tilde{p} such that, for all i and all $q_{-i} \in \tilde{X}_{-i}$, $\tilde{u}_i(\tilde{p}_i, q_{-i}) \geq \bar{u}_i$.

In other words, a game is safe if each player has a “good” equilibrium strategy. Clearly, any game with a unique mixed strategy Nash equilibrium is both equilibrium interchangeable and safe. Every two-player zero-sum game is both equilibrium interchangeable and safe. Aumann (1961) defines *almost strictly competitive* games, a class of two-player strategic form games generalizing two-player zero-sum games, that are both equilibrium interchangeable and safe.⁸ And any game with a dominant strategy equilibrium is safe (but not necessarily equilibrium interchangeable). The next theorem, the proof of which is self-evident, establishes the sufficiency of these conditions for equilibrium safety.

Theorem 13 *If Γ is equilibrium interchangeable or safe then it is equilibrium safe.*

It is clear that a safe game need not be equilibrium interchangeable, and the next example establishes the converse. Thus, the conditions are logically independent.

⁸For any bimatrix game (A_1, A_2) , where A_1 gives player 1’s payoffs and A_2 player 2’s, a *twisted equilibrium* is an equilibrium of the “twisted game” $(-A_2, -A_1)$. Aumann defines a bimatrix game as *almost strictly competitive* if (i) the set of equilibrium payoffs coincides with the set of twisted equilibrium payoffs, and (ii) the sets of equilibria and twisted equilibria have non-empty intersection.

Example 10 Let $|I| = 2$, $X_1 = \{x_1, x_2, x_3, x_4, x_5\}$, and $X_2 = \{y_1, y_2, y_3, y_4, y_5\}$, with payoffs as below.

	y_1	y_2	y_3	y_4	y_5
x_1	(0,0)	(1,-1)	(-1,1)	(-2,0)	(2,0)
x_2	(-1,1)	(0,0)	(1,-1)	(1,0)	(-1,0)
x_3	(1,-1)	(-1,1)	(0,0)	(1,0)	(-1,0)
x_4	(0,-2)	(0,1)	(0,1)	(-2,-1)	(-1,-1)
x_5	(0,2)	(0,-1)	(0,-1)	(-1,-1)	(-1,-2)

Note that the pair $p = (p_1, p_2)$, where $p_1 = p_2 = (1/3, 1/3, 1/3, 0, 0)$ is an equilibrium with zero payoffs for the players. Moreover, these strategies guarantee the players at least this payoff. Now define $p'_1 = p'_2 = (0, 0, 0, 1/2, 1/2)$, and note that (p_1, p'_2) and (p'_1, p_2) are equilibria but that (p'_1, p'_2) is not. Thus, this game is not equilibrium interchangeable. It remains only to be checked that zero is the unique equilibrium payoff.

Take any equilibrium $q = (q_1, q_2)$, and define $r_1 = q_1(x_1) + q_1(x_2) + q_1(x_3)$ and $r_2 = q_2(y_1) + q_2(y_2) + q_2(y_3)$. If $r_1 = r_2 = 1$, the players are essentially playing a symmetric zero-sum game, so their payoff must be zero. If $r_1 < 1$ and $r_2 = 1$, player 1's expected payoff from x_4 and x_5 is zero, so his payoff from q is zero. Thus, $q_2 = p_2$ (otherwise player 1 could deviate profitably) and 2's payoff is zero. Similarly if $r_1 = 1$ and $r_2 < 1$. If $r_1 < 1$ and $r_2 < 1$, then player 1's payoff from x_4 and x_5 is negative, worse than p_1 , contradicting our assumption that q is an equilibrium.

Theorem 13 allows us to apply the results of Kats and Thisse (1992) on equilibrium interchangeability. Their analysis uses the following more primitive conditions, defined

for strategic form games with possibly infinite strategy sets. The first condition is defined only for two-player games and the second extends it to multi-player games.

Definition 7

(i) A two-player game is **strictly competitive** if, for all i and $j \neq i$ and for all $x, y \in X$,

$$u_i(y_i, x_j) > u_i(x) \Leftrightarrow u_j(y_i, x_j) < u_j(x).$$

(ii) A strategic form game is **unilaterally competitive** if, for all i , all $x_i, y_i \in X_i$, and all $x_{-i} \in X_{-i}$,

$$(u_i(y_i, x_{-i}) > u_i(x)) \Leftrightarrow (\forall j \neq i)(u_j(y_i, x_{-i}) < u_j(x_{-i})).$$

(iii) A strategic form game is **weakly unilaterally competitive** if, for all i , all $x_i, y_i \in X_i$, and all $x_{-i} \in X_{-i}$,

$$(u_i(y_i, x_{-i}) > u_i(x)) \Rightarrow (\forall j \neq i)(u_j(y_i, x_{-i}) \leq u_j(x))$$

and

$$(u_i(y_i, x_{-i}) = u_i(x)) \Rightarrow (\forall j \neq i)(u_j(y_i, x_{-i}) = u_j(x)).$$

Thus, a two-player game is strictly competitive if its payoffs are strictly Pareto optimal. The idea of unilateral competitiveness extends this concept, but only applies the Pareto optimality criterion to unilateral changes in strategies. The third condition weakens unilateral competitiveness, now allowing for one player to improve his payoff with a unilateral move, as long as no other player is made better off. The next theorem follows directly from Theorem 13 and Kats and Thisse's Theorem 2, where they prove the sufficiency of their conditions for equilibrium interchangeability.

Theorem 14 *Let Γ be a finite strategic form game.*

(i) *If $|I| = 2$ and the mixed extension $\tilde{\Gamma}$ is weakly unilaterally competitive, then Γ is equilibrium safe.*

(ii) *If the mixed extension $\tilde{\Gamma}$ is unilaterally competitive then Γ is equilibrium safe.*

Unfortunately, this result uses conditions on the mixed extension of Γ , which may be difficult to verify. The next example shows that, even in two-player games, requiring strict competitiveness of Γ itself is not sufficient for uniqueness of the R -set or S^* -set.

Example 11 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c, d\}$, with payoffs below.

	a	b	c	d
a	(5,1)	(1,5)	(2,2)	(2,2)
b	(1,5)	(5,1)	(2,2)	(2,2)
c	(2,2)	(2,2)	(5,1)	(1,5)
d	(2,2)	(2,2)	(1,5)	(5,1)

This game is strictly competitive, but not equilibrium safe, as evidenced by the fact that it has two R -sets (and S^* -sets), $\{a, b\} \times \{a, b\}$ and $\{c, d\} \times \{c, d\}$.

The game in Example 11 does have a unique S -set and B -set, as we will see is true of all strictly competitive games.

6.3 Order Equivalent and Best Response Equivalent Games

Two games, $\Gamma = (I, (X_i)_{i \in I}, (u_i)_{i \in I})$ and $\Gamma' = (I', (X'_i)_{i \in I}, (u'_i)_{i \in I})$, are *order equivalent* if $I = I'$; for all i , $X_i = X'_i$; and for all i , all $x_i, y_i \in X_i$, and all $x_{-i} \in X_{-i}$,

$$u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i}) \Leftrightarrow u'_i(x_i, x_{-i}) \geq u'_i(y_i, x_{-i}).$$

In a two-player matrix game, for example, order equivalence means that the row player's ordering of cells in any given column is the same, and the column player's ordering of cells in any given row is the same. Relationships between payoffs in cells that do not lie on the same row or column are unrestricted.

Two games, Γ and Γ' , are *best response equivalent* if $I = I'$; for all i , $X_i = X'_i$; and for all i and all $p_{-i} \in \tilde{X}_{-i}$, i 's pure strategy best responses to p_{-i} in Γ and Γ' coincide. Abusing notation slightly, $BR_i(p_{-i}) = BR'_i(p_{-i})$. Two games may be order equivalent but not best response equivalent, and they may be best response equivalent

but not order equivalent.⁹ It is clear that R -solutions are invariant under best response equivalent transformations: if Γ and Γ' are best response equivalent, Y is a R -solution of Γ if and only if Y is a R -solution of Γ' . This gives us the following extension of Theorem 12.

Theorem 15 *If Γ' is best response equivalent to an equilibrium safe game Γ , then Γ' has a unique R -, S^* -, B -, and S -set.*

Note that S -solutions are invariant with respect to order equivalent transformations — if Γ and Γ' are order equivalent, Y is a S -solution of Γ if and only if Y is a S -solution of Γ' — and that the same is true of B -solutions. Thus, we can extend Theorem 12 even further for S -sets and B -sets.

Theorem 16 *If Γ'' is order equivalent to Γ' , and Γ' is best response equivalent to an equilibrium safe game Γ , then Γ'' has a unique B -set and a unique S -set.*

Theorem 16 immediately yields uniqueness of the S -set and B -set in two-player strictly competitive games, which are known to be order equivalent to zero-sum games. The next example illustrates the scope of the theorem.

Example 12 Let $|I| = 2$ and $X_1 = X_2 = \{a, b, c, d\}$, with payoffs below.

	a	b	c	d
a	(5,1)	(1,5)	(3,3)	(3,3)
b	(1,5)	(5,1)	(3,3)	(3,3)
c	(3,3)	(3,3)	(5,1)	(1,5)
d	(3,3)	(3,3)	(1,5)	(5,1)

⁹See Rosenthal's (1974) Examples 5 and 6.

This game is order equivalent (but not best response equivalent) to the game in Example 11. It is equilibrium safe, since $p_1 = p_2 = (1/2, 1/2, 0, 0)$ is a safe equilibrium, and has a unique R -set and unique S -set. Thus, as noted above, the game in Example 11 has a unique S -set and B -set as well.

We have proved elsewhere that the S -set is unique in two-player weakly unilaterally competitive games, but we have not determined whether this result can be deduced from Theorem 16. That is, we do not know if every two-player weakly unilaterally competitive game is order equivalent to (a game that is best response equivalent to) an equilibrium safe game. We do know that there are two-player weakly unilaterally competitive games that cannot be obtained from strictly competitive games via this path. The game in Example 8 illustrates this. Because it is a two-by-two game, best response equivalence buys no more than order equivalence. Note that, in any order equivalent game, row player will prefer (a, a) to (b, a) and (b, b) to (a, b) ; and column player will be indifferent between (a, a) and (a, b) and between (b, a) and (b, b) . If the equivalent game is strictly competitive, the column player must prefer (b, a) to (a, a) and (a, b) to (b, b) , but then the column player's preferences are intransitive, a contradiction.

7 Infinite Games

In this section we explore the existence of Q -solutions in infinite games, where we allow I to be an arbitrary (possibly infinite) set of players and view each X_i as a topological space, with X and X_{-i} endowed with the product topologies. Though our above analysis was restricted to finite games, the definition of dominance structure extends to the infinite setting without modification. We now extend our earlier definition of Q -solution, recalling Proposition 2: for finite games and transitive Q , Y is a Q -solution if and only if Y_i is a minimal subset of X_i possessing the outer solution property with respect to $Q_i(Y_{-i})$. Of course, each Y_i is trivially compact in finite games.

Definition 8 *A set Y is a Q -solution if, for all i , Y_i is minimal among the compact subsets of X_i possessing the outer solution property with respect to $Q_i(Y_{-i})$.*

The main conceptual modifications we make in extending our concept to general games are to replace the inner solution property with minimality and to restrict choice

sets to be compact. The next proposition shows that, under transitivity and an appropriate continuity condition, minimality amounts to the inner solution property. We say Q is *upper semi-continuous* if, for all i , all compact Y_{-i} , and all $x_i \in X_i$, the set $\{y_i \in X_i | x_i Q_i(Y_{-i})y_i\}$ is open.

Proposition 12 *Assume X_i is compact, Q is transitive and upper semi-continuous, and Y_{-i} is compact. The set Y_i is minimal among the compact sets possessing the outer solution property with respect to $Q_i(Y_{-i})$ if and only if Y_i possesses the solution property with respect to $Q_i(Y_{-i})$.*

Shapley dominance is an example of an upper semi-continuous dominance structure, under usual assumptions.

Proposition 13 *If each u_i is continuous then S is upper semi-continuous.*

Propositions 12 and 13 show that the restriction to compact choice sets is, in one sense, not binding for Shapley dominance: a S -solution under the current definition is an S -solution under the definition of Section 2. The next example shows that the restriction is binding, in another sense: there are S -solutions under the definition of Section 2 that are not S -solutions under the new definition, because choice sets of the players are non-compact.

Example 13 Let $|I| = 2$, $X_1 = X_2 = [0, 1]$, with the following payoffs

$$u_i(x_1, x_2) = \begin{cases} \max\{0, (x_1 - 1/2)x_2 - x_1 + 1/2\} & \text{if } 0 \leq x_1 \leq 1/2 \\ (x_1 - 1/2)|x_2 - 1/2| & \text{if } 1/2 \leq x_1 \leq 1 \end{cases}$$

for player 1 and symmetric payoffs for player 2. For $x_1 \leq 1/2$, player 1's payoffs are downward sloping in x_2 until they hit zero; for $x_1 = 1/2$, player 1's payoffs are constant at zero for all x_2 ; for $x_1 \geq 1/2$, player 1's payoffs are increasing with distance from $1/2$. Here, $Y_1 \times Y_2$ is a S -solution (as defined in Sections 2 and 3), where $Y_1 = Y_2 = [0, 1/2) \cup \{1\}$, but these sets are clearly not compact. Note that, for player 1, every $x_1 \notin Y_1$ is dominated, in the Shapley sense, over Y_2 by $y_1 = 1$. No strategy in Y_1 is dominated over Y_2 .

We next prove existence of Q -solutions, as just defined, for general games. In view of our previous comments, existence of an S -solution under the new definition yields an

S -solution under the definition of Section 3. Rather than imposing continuity conditions directly on the payoff functions of the players, we work with the dominance structure Q . Our continuity condition combines certain aspects of upper semi-continuity and transitivity. A dominance structure Q is *transitive-continuous* if, for all $i \in I$, all $x_i, y_i, z_i \in X_i$, all nets $x_i^\alpha \rightarrow x_i$, and all compact subsets $Y_{-i} \subseteq X_{-i}$,

$$(y_i Q_i(Y_{-i}) z_i) \wedge (\forall \alpha)(x_i^\alpha Q_i(Y_{-i}) y_i) \Rightarrow (x_i Q_i(Y_{-i}) z_i).$$

In finite games, transitive-continuity reduces to transitivity. In general, transitive-continuity implies transitivity, while transitivity and upper semi-continuity together imply transitive-continuity.

Proposition 14 *Let Q be transitive-continuous and monotonic. Consider a compact outer Q -solution Z and an inner Q -core Z' . There exists a (minimal) Q -solution Y such that $Z' \subseteq Y \subseteq Z$.*

Transitive-continuity does not, however, imply upper semi-continuity: neither W nor N are generally upper semi-continuous, but both, in addition to S , are transitive-continuous under weak topological assumptions on payoff functions. Note that B is transitive-continuous regardless of the continuity properties of payoff functions. We say u_i is *upper semi-continuous in x_i* if, for all $x_{-i} \in X_{-i}$, the set $\{y_i \in X_i | u_i(x) > u_i(y_i, x_{-i})\}$ is open.

Proposition 15 *If each u_i is upper semi-continuous in x_i then S , W , and N are transitive-continuous.*

Propositions 14 and 15 immediately imply the existence of S -solutions and N -solutions in general games. Unlike existence results for Nash equilibrium, no convexity assumptions are needed. The automatic transitive-continuity of B suggests an modification of the dominance structure Q in discontinuous games for which existence is guaranteed.

(14) **Flat Q** Define $x_i \overline{Q}_i(Y_{-i}) y_i$ if and only if $z_i Q_i(Y_{-i}) y_i$ for some $z_i \in X_i$.

If Q is monotonic then so is \overline{Q} . It is transitive-continuous as well, so \overline{Q} -solutions always exist. The next example shows, however, that, we don't obtain something for

nothing: when S -solutions don't exist, the predictions of \bar{S} -solutions are less than satisfactory.

Example 14 Let $|I| = 2$, $X_1 = X_2 = [0, 1]$, with the following payoffs

$$u_1(x_1, x_2) = \begin{cases} -x_1x_2 & \text{if } x_1 < 1 \\ -2 & \text{if } x_1 = 1, \end{cases}$$

for player 1 and symmetric payoffs for player 2. Here, the unique \bar{S} -solution is $[0, 1] \times [0, 1]$. It is not an S -solution because $[0, 1]$ does not have the inner solution property with respect to $S_1([0, 1])$ or $S_2([0, 1])$. In fact, $x_i = 1$ is strictly dominated for both players.

A Proofs of Propositions

Proposition 1 *Assume Q_i is irreflexive and transitive. A set $Y_i \subseteq X_i$ has the solution property with respect to Q_i if and only if it has the core property with respect to Q_i .*

Proof: Assume Y_i has the solution property and take $x_i \in Y_i$. To see that $(\forall y_i \in X_i \setminus \{x_i\})(\neg y_i Q_i x_i)$, take any $y_i \in X_i \setminus Y_i$, and suppose $y_i Q_i x_i$. Since $y_i \notin Y_i$, the solution property implies the existence of $z_i \in Y_i$ such that $z_i Q_i y_i$. Then transitivity implies $z_i Q_i x_i$ and irreflexivity implies $z_i \neq x_i$, but this contradicts the assumption that $x_i \in Y_i$. Now take any x_i such that $(\forall y_i \in X_i)(\neg y_i Q_i x_i)$. This clearly implies $(\forall y_i \in Y_i)(\neg y_i Q_i x_i)$, so the outer solution property yields $x_i \in Y_i$.

Assume Y_i has the core property and take $x_i \in Y_i$. Clearly, $(\forall y_i \in Y_i)(\neg y_i Q_i x_i)$. Now take any x_i such that $(\forall y_i \in Y_i)(\neg y_i Q_i x_i)$. If $x_i \notin Y_i$, the core property implies there exists $y_i \in X_i \setminus Y_i$ such that $y_i Q_i x_i$. Since X_i is finite, irreflexivity and transitivity of Q_i imply the existence of a strategy z_i such that $z_i Q_i x_i$ and, for all $w_i \in X_i \setminus \{z_i\}$, $\neg w_i Q_i z_i$. Then the core property yields $z_i \in Y_i$, as desired. ■

Proposition 2 *If Y is a Q -solution then, for all i , Y_i is a minimal subset of X_i possessing the outer solution property with respect to $Q_i(Y_{-i})$. If Q is transitive, the converse is also true.*

Proof: Let Y be a Q -solution, and take any i and $Z_i \subset \subset Y_i$ such that Z_i has the outer solution property with respect to $Q_i(Y_{-i})$. Take $x_i \in Y_i \setminus Z_i$, and note that, by the outer solution property, there exists $y_i \in Z_i \subseteq Y_i$ such that $y_i Q_i(Y_{-i})x_i$. But then, since Y_i has the inner solution property, $x_i \notin Y_i$, a contradiction.

Suppose that, for all i , Y_i is a minimal subset possessing the outer solution property with respect to $Q_i(Y_{-i})$. Take $x_i, y_i \in Y_i$ and suppose $y_i Q_i(Y_{-i})x_i$. Then $Y_i \setminus \{x_i\}$ has the outer solution property, contradicting minimality. To see this, note that, for each $z_i \notin Y_i \setminus \{x_i\}$, either $z_i = x_i$ or there exists some $w_i \in Y_i$ such that $w_i Q_i(Y_{-i})z_i$. If $z_i = x_i$ then y_i fulfills the definition of the outer solution property. If not, then either $w_i = x_i$ or $w_i \neq x_i$. If $w_i = x_i$ then transitivity implies $y_i Q_i(Y_{-i})z_i$, fulfilling the definition of the outer solution property. Finally, if $w_i \neq x_i$ then w_i itself suffices. Therefore, Y_i has the inner (as well as the outer) solution property. ■

Proposition 3 *Let Q be transitive and monotonic. Consider an outer Q -solution Z and an inner Q -core Z' . If Y is minimal among the outer Q -solutions satisfying $Z' \subseteq Y \subseteq Z$, then Y is a Q -solution.*

Proof: Suppose there exist i and $Z_i \subset\subset Y_i$ such that Z_i has the outer solution property with respect to $Q_i(Y_{-i})$. Take $x_i \in Z'_i$ and suppose $x_i \notin Z_i$. By the outer solution property, there exists $z_i \in Z_i$ such that $z_i Q_i(Y_{-i})x_i$, and monotonicity implies $z_i Q_i(Z'_i)x_i$. But this contradicts $x_i \in Z'_i$, and we conclude that $Z' \subseteq Z_i \times Y_{-i} \subseteq Z$. To see that $Z_i \times Y_{-i}$ is an outer Q -solution, consider $j \neq i$ and take $x_j \notin Y_j$. By the solution property with respect to $Q_j(Y_{-j})$, there exists $y_j \in Y_j$ such that $y_j Q_j(Y_{-j})x_j$, and monotonicity yields $y_j Q_j(Z_i \times Y_{-i,j})x_j$. This contradicts the minimality of Y . Therefore, there do not exist such i and Z_i , and Proposition 2 implies that Y is a Q -solution, as desired. ■

Proposition 4 *If Q is transitive, monotonic, and hard then the maximal Q -solution is unique.*

Proof: Let Y and Z be any two Q -solutions, and, for each i , define $Z'_i = Y_i \cup Z_i$, and let $Z' = \times_{i \in I} Z'_i$. To see that Z' is an inner Q -core, take $x_i \in Z'_i$ and suppose $y_i Q_i(Z'_i)x_i$ for some $y_i \neq x_i$. Assuming without loss of generality that $x_i \in Y_i$, monotonicity implies $y_i Q_i(Y_{-i})x_i$, contradicting the inner core property with respect to $Q_i(Y_{-i})$. Thus, Proposition 3 implies the existence of a Q -solution Y' such that $Z' \subseteq Y'$. The desired result follows, since $Y, Z \subseteq Z' \subseteq Y'$. ■

Proposition 5 *If Q is transitive, monotonic, and hard then $U^Q(X)$ is the maximal Q -solution.*

Proof: Let Y be the maximal Q -solution, unique by Proposition 4. Since Q is hard, each Y_i has the inner core property with respect to $Q_i(Y_{-i})$. This implies $Y \subseteq U^{Q,1}(X)$, for $x_i \in Y_i$ and $y_i Q_i(X_{-i})x_i$ would imply, by monotonicity, $y_i Q_i(Y_{-i})x_i$, a contradiction. Take any $k \geq 1$ satisfying the induction hypothesis: for all $l < k$, $Y \subseteq U^{Q,l}(X)$. Suppose there exist some i and $x_i \in Y_i \setminus U^{Q,k}_i(X)$, so there exists $y_i \in X_i$ such that $y_i Q_i(U^{Q,k-1}_{-i}(X))x_i$. By monotonicity and the hypothesis that $Y_{-i} \subseteq U^{Q,k-1}_{-i}(X)$, $y_i Q_i(Y_{-i})x_i$, contradicting the inner core property. Therefore, $Y \subseteq U^Q(X)$. Since $U^Q(X)$ is clearly an inner Q -core, Proposition 3 implies the existence of a Q -solution $Z \supseteq U^Q(X)$. Then $Y \subseteq U^Q(X) \subseteq Z \subseteq Y$ implies $Y = U^Q(X)$. ■

Proposition 6 *Let Q be a transitive, monotonic, hard dominance structure. Let Y^1, Y^2, \dots, Y^K be a sequence of product sets decreasing to an inner Q -core Y^K such that $Y^1 = X$, and for all k and all i , Y^k is an outer $Q_i(Y^{k-1}_{-i})$ -core. Then $Y^K = U^Q(X)$.*

Proof: Since Y^K is an inner Q -core, Proposition 3 yields the existence of a Q -solution $Y \supseteq Y^K$. By Proposition 5, the unique maximal Q -solution is $U^Q(X)$, so $Y^K \subseteq Y \subseteq U^Q(X)$. It remains to show that $U^Q(X) \subseteq Y^K$. Clearly, $U^Q(X) \subseteq Y^1 = X$. Take any $k \geq l$ satisfying the induction hypothesis: for all $l < k$, $U^Q(X) \subseteq Y^l$. Suppose there exists some i and $x_i \in U^Q(X) \setminus Y^k_i$. Since Y^k_i is an outer $Q_i(Y^{k-1}_{-i})$ -core, there exists $y_i \in X_i$ such that $y_i Q_i(Y^{k-1}_{-i})x_i$. By the induction hypothesis, $U^Q_{-i}(X) \subseteq Y^{k-1}_{-i}$, so monotonicity implies $y_i Q_i(U^Q_{-i}(X))x_i$, a contradiction. ■

Proposition 7 *If Q is weakly irreflexive, transitive, and, for all i and all $Y_{-i} \subseteq X_{-i}$, the $Q_i(Y_{-i})$ -core is non-empty, then Q is hard.*

Proof: Let Y be a Q -solution, and suppose that $x_i Q_i(Y_{-i}) y_i$ for some i , some $x_i \notin Y_i$, and some $y_i \in Y_i$. By the outer solution property with respect to $Q_i(Y_{-i})$, there exists $z_i \in Y_i$ such that $z_i Q_i(Y_{-i}) x_i$, and transitivity implies $z_i Q_i(Y_{-i}) y_i$. By the inner solution property with respect to $Q_i(Y_{-i})$, $z_i = y_i$. Then weak irreflexivity implies that $w_i Q_i(Y_{-i}) z_i$ for all $w_i \in X_i$. Let w_i be an element of the $Q_i(Y_{-i})$ -core, which is non-empty by assumption. Therefore, $w_i \in Y_i$ and $w_i \neq z_i$, contradicting the inner solution property with respect to $Q_i(Y_{-i})$. ■

Proposition 8 *Let Q be a dominance structure for Γ .*

- (i) Q^* is transitive and weakly irreflexive.
- (ii) If Q is monotonic then Q^* is monotonic.
- (iii) If, for all i and all $Y_{-i} \subseteq X_{-i}$, the $\tilde{Q}_i(Y_{-i})$ -core is non-empty, then Q^* is hard.

Proof: (i) Transitivity and weak irreflexivity are obvious. (ii) Monotonicity of Q^* follows from monotonicity of \tilde{Q} . (iii) Let x_i be an element of the $\tilde{Q}_i(Y_{-i})$ -core for some Q^* -solution Y , so $x_i \in Y_i$. If $z_i Q_i^*(Y_{-i}) y_i$ for some $y_i \in Y_i$ and $z_i \in X_i$, we have $y_i \neq x_i$ and $x_i Q_i^*(Y_{-i}) y_i$, contradicting the inner solution property with respect to $Q_i^*(Y_{-i})$. ■

Proposition 9 *Assume Q is stratified. The set Y is a Q^* -solution if and only if the following two conditions hold.*

- (i) For all i and all $x_i \notin Y_i$, there exists $p_i \in \tilde{Y}_i$ such that $p_i \tilde{Q}_i(Y_{-i}) x_i$.
- (ii) For all i and all $x_i \in Y_i$, there is no $p_i \in \tilde{Y}_i$ such that $p_i \tilde{Q}_i(Y_{-i}) x_i$.

Proof: Suppose Y is a Q^* -solution. To deduce (i), suppose $x_i \notin Y_i$, so there exists $y_i \in Y_i$ such that $y_i Q_i^*(Y_{-i}) x_i$. Thus, there exists $p_i \in \tilde{X}_i$ such that $p_i \tilde{Q}_i(Y_{-i}) x_i$. Because Q is stratified, there exists $\tilde{p}_i \in \tilde{X}_i$ such that $\tilde{p}_i \tilde{Q}_i(Y_{-i}) x_i$ and $\sigma_i(\tilde{p}_i)$ is an inner $Q_i^*(Y_{-i})$ -core. This implies that $\sigma_i(\tilde{p}_i) \subseteq Y_i$, or equivalently, that $\tilde{p}_i \in \tilde{Y}_i$. To deduce (ii), take $x_i \in Y_i$ and suppose $p_i \tilde{Q}_i(Y_{-i}) x_i$ for some $p_i \in \tilde{X}_i$. Because Q is stratified, we may assume that $\sigma_i(p_i)$ is an inner $Q_i^*(Y_{-i})$ -core. Thus, $\sigma_i(p_i) \subseteq Y_i \setminus \{x_i\}$ and there exists $y_i \in Y_i \setminus \{x_i\}$ such that $y_i Q_i^*(Y_{-i}) x_i$, a contradiction.

Now suppose (i) and (ii) hold. Clearly, Y is an outer Q^* -solution. To see it is an inner Q^* -solution, take $x_i, y_i \in Y_i$, suppose $x_i Q_i^*(Y_{-i}) y_i$, and let $p_i \in \tilde{X}_i$ satisfy $p_i \tilde{Q}_i(Y_{-i}) y_i$. Because Q is stratified, we may assume $p_i \in \tilde{Y}_i$, contradicting (ii). ■

Proposition 10 *Assume Q is weakly irreflexive, transitive, and monotonic. If Y and Z are distinct Q -sets, then $Y \cap Z = \emptyset$.*

Proof: If $Y \cap Z \neq \emptyset$ then, because Q is monotonic, $Y \cap Z$ is an outer Q -solution. To see this, take i and $x_i \notin Y_i \cap Z_i$, say $x_i \notin Y_i$. Then there exists $y_i^1 \in Y_i$ such that $y_i^1 Q_i(Y_{-i}) x_i$, and, by monotonicity, $y_i^1 Q_i(Y_{-i} \cap Z_{-i}) x_i$. If $y_i^1 \in Z_i$, we are done. If not, there exists $y_i^2 \in Z_i$ such that $y_i^2 Q_i(Y_{-i} \cap Z_{-i}) y_i^1$, and so on. If this sequence does not lead us to $z_i \in Y_i \cap Z_i$, finiteness of X_i yields k and l such that $y_i^k = y_i^l$. By transitivity, $y_i^k Q_i(Y_{-i} \cap Z_{-i}) y_i^k$, and weak irreflexivity then implies $z_i Q_i(Y_{-i} \cap Z_{-i}) y_i^k$. A last application of transitivity gives $z_i Q_i(Y_{-i} \cap Z_{-i}) x_i$. Thus, $Y \cap Z$ is an outer Q -solution, so Proposition 3 yields a Q -set $Y' \subset Y \cap Z$, contradicting the minimality of Y and Z . ■

Proposition 11 *Let Q' be transitive and monotonic. If Q is heavier than Q' and the Q' -set is unique, then the Q -set is unique.*

Proof: Suppose Y and Z are distinct Q -sets. By Proposition 10, $Y \cap Z = \emptyset$. But Proposition 3 yields Q' -sets $Y' \subseteq Y$ and $Z' \subseteq Z$, a contradiction. ■

Proposition 12 *Assume X_i is compact, Q is transitive and upper semi-continuous, and Y_{-i} is compact. The set Y_i is minimal among the compact sets possessing the outer solution property with respect to $Q_i(Y_{-i})$ if and only if Y_i possesses the solution property with respect to $Q_i(Y_{-i})$.*

Proof: Let Y_i be minimal among the compact sets possessing the outer solution property with respect to $Q_i(Y_{-i})$. Suppose that $y_i Q_i(Y_{-i}) x_i$ for distinct $x_i, y_i \in Y_i$, and let $G = \{z_i \in Y_i \mid y_i Q_i(Y_{-i}) z_i\}$. By upper semi-continuity, $Y_i \setminus G$ is compact, and we claim it has the outer solution property with respect to $Q_i(Y_{-i})$. For each $z_i \notin Y_i \setminus G$, either $z_i \in G$ or there exists some $w_i \in Y_i$ such that $w_i Q_i(Y_{-i}) z_i$. If $z_i \in G$ then y_i fulfills the definition of the outer solution property. If not, then either $w_i \in G$ or $w_i \in Y_i \setminus G$. If $w_i \in G$ then $y_i Q_i(Y_{-i}) w_i Q_i(Y_{-i}) x_i$, and transitivity implies $y_i Q_i(Y_{-i}) x_i$, as desired. If $w_i \in Y_i \setminus G$ then w_i itself fulfills the outer solution property. Therefore, Y_i has the solution property with respect to $Q_i(Y_{-i})$.

Now let Y_i have the solution property with respect to $Q_i(Y_{-i})$. Take i and $x_i^\alpha \rightarrow x_i$, where $x_i^\alpha \in Y_i$ for all α , and suppose $x_i \notin Y_i$. By the outer solution property with respect to $Q_i(Y_{-i})$, there exists $y_i \in Y_i$ such that $y_i Q_i(Y_{-i}) x_i$. Since Q is upper semi-continuous there exists an open set G with $x_i \in G$ such that, for all $z_i \in G$, $y_i Q_i(Y_{-i}) z_i$. But then $y_i Q_i(Y_{-i}) x_i^\alpha$ for some α , contradicting the inner solution property with respect to $Q_i(Y_{-i})$. Therefore, Y_i is compact. If it is not minimal among the compact sets possessing the outer solution property with respect to $Q_i(Y_{-i})$, there is a smaller such set $Z_i \subseteq Y_i$ and $x_i \in Y_i \setminus Z_i$. By the outer solution property, there exists $y_i \in Z_i$ such that $y_i Q_i(Y_{-i}) x_i$, but this contradicts the assumption that Y_i has the inner solution property with respect to $Q_i(Y_{-i})$. ■

Proposition 13 *If each u_i is continuous then S is upper semi-continuous.*

Proof: Let Y_{-i} be compact and suppose $y_i S_i(Y_{-i}) x_i$, so, for all $x_{-i} \in Y_{-i}$, $u_i(y_i, x_{-i}) - u_i(x) > 0$. Since Y_{-i} is compact and u_i is continuous, $\min_{x_{-i} \in Y_{-i}} u_i(y_i, x_{-i}) - u_i(z_i, x_{-i})$ is well-defined for all $z_i \in X_i$ and, by Berge's theorem, continuous as a function of z_i . Furthermore, $\min_{x_{-i} \in Y_{-i}} u_i(y_i, x_{-i}) - u_i(x) = \epsilon > 0$. Therefore, there exists an open set G with $x_i \in G$ such that $\min_{x_{-i} \in Y_{-i}} u_i(y_i, x_{-i}) - u_i(z_i, x_{-i}) > 0$ for all $z_i \in G$. In other words, $y_i S_i(Y_{-i}) z_i$ for all $z_i \in G$. ■

Proposition 14 *Let Q be transitive-continuous and monotonic. Consider a compact outer Q -solution Z and an inner Q -core Z' . There exists a (minimal) Q -solution Y such that $Z' \subseteq Y \subseteq Z$.*

Proof: Let \mathcal{Y} be the collection of compact outer Q -solutions included in Z . (This set is non-empty since $Z \in \mathcal{Y}$.) Given a chain $\mathcal{C} \subseteq \mathcal{Y}$ totally ordered by set-inclusion, we claim that $\bigcap \mathcal{C} \in \mathcal{Y}$. First, note that \mathcal{C} has the finite intersection property, so, because X is compact, $\bigcap \mathcal{C} \neq \emptyset$. Next, note that $\bigcap \mathcal{C}$ is a compact product set, which we write $Y^* = \prod_{i \in I} Y_i^*$. Now, note that Y^* is an outer Q -solution. To see this, take $i \in I$ and $z_i \notin Y_i^*$. Since $z_i \notin Y_i^*$, there exists $Y' \in \mathcal{C}$ such that $z_i \notin Y'_i$. Because Y'_i has the outer solution property with respect

to $Q_i(Y'_{-i})$, there exists $y_i \in Y'_i$ such that $y_i Q_i(Y'_{-i}) z_i$ and, by monotonicity, $y_i Q_i(Y^*_{-i}) z_i$. If $y_i \notin Y_i^*$, there exists $Y'' \in \mathcal{C}$ such that $y_i \notin Y''_i$. For each $Y \in \mathcal{C}$ included in Y'' , $y_i \notin Y_i$, so the outer solution property with respect to $Q_i(Y_{-i})$ implies the existence of $x_i^Y \in Y_i$ such that $x_i^Y Q_i(Y_{-i}) y_i$. By monotonicity, $x_i^Y Q_i(Y^*_{-i}) y_i$. Since X_i is compact, the net $\{x_i^Y\}_{Y \subseteq Y''}$ has a limit point $x_i^* \in Y_i^*$. By transitive-continuity of Q , we have $x_i^* Q_i(Y^*_{-i}) z_i$. Thus, Y_i^* has the outer solution property with respect to $Q_i(Y^*_{-i})$. By Zorn's lemma, the collection \mathcal{Y} contains a minimal element, which we denote Y^{**} . Finally, note that Y_i^{**} is minimal among the compact subsets of X_i possessing the outer solution property with respect to $Q_i(Y^{**}_{-i})$. To see this, suppose $\hat{Y}_i \subset\subset Y_i^{**}$ has the outer solution property with respect to $Q_i(Y^{**}_{-i})$. Invoking monotonicity of Q , $\hat{Y}_i \times Y^{**}_{-i} \in \mathcal{Y}$, is an outer Q -solution. Indeed, if $z_i \notin Y_i^{**}$, there exists $y_i \in Y_i^{**}$ such that $y_i Q_i(Y^{**}_{-i}) z_i$, and, since Z' is an inner Q -core, we conclude that $Z' \subseteq \hat{Y}_i \times Y^{**}_{-i}$, contradicting the minimality of Y^{**} . Therefore, Y^{**} is a Q -solution. ■

Proposition 15 *If each u_i is upper semi-continuous in x_i then S , W , and N are transitive-continuous.*

Proof: Let $x_i, y_i, z_i, \{x_i^\alpha\}$, and Y_{-i} be as in the definition of transitive-continuity. For all $y_{-i} \in Y_{-i}$, the net $\{x_i^\alpha\}$ lies in i 's weak upper contour set at y_i , i.e. $x_i^\alpha \in \{w_i \in X_i | u_i(w_i, y_{-i}) \geq u_i(y)\}$ for all α . This set is closed by upper semi-continuity, so $u_i(x_i, y_{-i}) \geq u_i(y)$. Then $y_i W_i(Y_{-i}) z_i$ implies $x_i W_i(Y_{-i}) z_i$. Similar arguments apply to N and S . ■

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Figure 1: Lattice structure of Q -solutions. A solid arrow from Q to Q' signifies that the Q -solutions are weaker in the full sense than the Q' -solutions. A dashed arrow indicates that every Q -solution includes a Q' -solution.