Equilibrium Equivalence under Expected Plurality and Probability of Winning Maximization*

INCOMPLETE: ONE SECTION MISSING

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*I owe Mark Fey for several very helpful discussions of equilibrium equivalence.

Abstract

This paper studies competition between two office-motivated candidates when voters vote probabilistically and candidates maximize expected plurality or they maximize probability of winning. Every symmetric equilibrium under probability of winning is an equilibrium under expected plurality, but the converse hinges on some subtle issues. In a standard model of probabilistic voting, equilibria under probability of winning — if any — must be symmetric and must occur at the utilitarian welfare maximizing policy position. If the utility functions of voters are negative definite at that point, then it is indeed a local equilibrium under probability of winning. If negative definiteness is weakened to strict concavity (a seemingly innocuous move), this implication breaks down, and pure strategy equilibria under probability of winning need not exist. This possibility is examined in detail.

1 Introduction

It as evident that electoral candidates, in running their campaigns, care about their probability of winning and about their expected margin of victory, i.e., their expected "plurality." Intuition suggests that the practical difference between these objectives in terms of the incentives they give candidates is slim, but there has been little work on the formal analysis of this intuition. From a technical point of view, expected plurality maximization is a nicely behaved objective function, and it has allowed a considerable amount of work on the existence and location of electoral equilibria in various models of voting behavior. Probability of winning, in contrast, is often much less manageable. If we can show these objective functions give rise to identical sets of equilibria under reasonable conditions, that would mean the analysis of elections could proceed under the assumption of expected plurality maximization with no loss of generality. If equivalence does not hold generally, that is equally important to know: the value of equilibrium results using expected plurality maximization cannot be assessed unless we understand how sensitive they are to the specification of candidate objective functions.

In this paper, we focus on competition between two candidates when voting behavior is probabilistic, i.e., the platforms adopted by the candidates determine a probability distribution over the votes of voters, and we consider this issue of "equilibrium equivalence." It is a simple matter to show that, under some regularity conditions on voting behavior, every symmetric equilibrium under expected plurality is an equilibrium under probability of winning. The converse hinges on some subtle issues, which we take up in a standard model of probabilistic voting, the "additive bias" model, satisfying our regularity conditions. Specifically, we assume that each voter has a utility function reflecting his/her policy preferences, known to the candidates, and that each voter receives a randomly distributed utility shock, reflecting the voter's preferences over non-policy (fixed) characteristics of the candidates. This shock is added to the policy utility from one of the candidate's platforms and is, for simplicity, assumed to be uniformly distributed. Banks and Duggan (1999) collect and unify many of the existing results for this model, including: an equilibrium exists, it is unique, and it is located at the utilitarian welfare maximizing policy position.

We show that equilibria under probability of winning — if any — must have both candidates adopting the utilitarian optimum. Thus, there is at most one such equilibrium, and it exhibits the feature that the candidates adopt identical platforms, a phenomenon called "policy coincidence" by Banks and Duggan (1999). This result generalizes Theorem 5 of Lindbeck and Weibull (1987), who consider a version of the additive bias model in which policies consist of distributions of resources across districts. It contrasts with Calvert's (1985) Theorem 4, which holds in a more general model of probabilistic voting but relies on the existence of an "estimated median" (an equilibrium possessing certain properties). As a consequence of our result, the converse question of equilibrium equivalence reduces to: Is it an equilibrium under probability of winning for both candidates to locate at the utilitarian optimum? We show that, if the voters' utility functions are negative definite at the optimum, then it is indeed a "local" equilibrium, i.e., there do not exist arbitrarily small deviations that will improve either candidate's probability of winning.

If negative definiteness is weakened even to strict concavity, however, that implication need not hold. When there are three voters, a necessary condition for the utilitarian optimum to fail to be a local equilibrium under probability of winning is that it not lie in the core, i.e., there is some policy platform preferred to the optimum by a majority of voters. A stronger condition, involving the second and third derivatives of the voters' utility functions, is sufficient for the equivalence to break down. We give an explicit example, with three voters, a one-dimensional policy space, and strictly concave utility functions, in which our sufficient condition holds: either candidate can increase his/her probability of winning by moving an arbitrarily small distance from the utilitarian optimum — in fact, by moving in a direction that is worse for a majority of voters. In the example, each voter's utility function is negative definite everywhere but at the welfare optimum, where their second derivatives are equal to zero.

We then extend these results to an arbitrary odd number of voters, finding one major difference: when the voters are greater than three in number, the existence of a platform majority-preferred to the utilitarian optimum is no longer necessary for the optimum to fail to be a local equilibrium under probability of winning. We give an explicit example, with five voters, a one-dimensional policy space, and strictly concave utility functions, demonstrating this. An important observation is that, in the cases where the optimum fails to be a local equilibrium (and therefore fails to be an equilibrium) under probability of winning, there is *no* equilibrium under probability of winning: as mentioned above, the only possible equilibrium is where both candidates locate at the utilitarian optimum. Thus, we demonstrate that equilibria under probability of winning do not exist generally in a standard model of probabilistic voting — indeed, one in which the candidates' strategy spaces are compact and convex and their payoff functions continuous.[1]

Interest in equivalence of objective functions dates back to early studies in positive political theory and was revived recently by Patty (1999a,b), who looks at a stronger notion of equivalence, called "best response equivalence," in one paper and considers some issues in equilibrium equivalence in the second. Patty (2000, Chapter 2) extends our arguments to a more general model of probabilistic voting and allows for any finite number of candidates. In earlier work, Aranson, Hinich, and Ordeshook (1974) state an equilibrium equivalence result for expected plurality and probability of winning maximization, but their model of probabilistic voting possesses the undesirable feature that vote probabilities may dip below zero or exceed one. Hinich and Ordeshook (1970) consider a different equivalence, that between expected plurality and expected vote maximization.

2 The Campaign Competition Game

We consider campaign competition between two candidates, A and B, for the votes of an odd number $n \geq 3$ of voters. Let N denote the set of voters. Let m denote a bare majority of voters, i.e., $m = (n+1)/2$. Let $X \subseteq \Re^d$ denote a set of possible policy platforms for the candidates. Candidates choose from X simultaneously, and each voter i then casts a vote $v_i \in \{A, B\}$, where

[1] Non-existence is in our examples is driven by non-convexities of the candidates' payoff functions. Ball (1999) gives an example of non-existence that exploits discontinuities in a slightly different model.

$v_i = A$ obviously denotes a vote for A and similarly for $v_i = B$. A pair (x_A, x_B) of platforms determines the distribution of the profile $(v_1, \ldots, v_n)$ of votes, which we view as a random variable. Let p_i denote the marginal probability that i votes for A, and let $p = (p_1, \ldots, p_n)$ denote the profile of marginal probabilities. The expected vote for A is

$$E_A(p) \;=\; \sum_{i \in N} p_i.$$

We assume throughout that the votes of the voters are independently distributed, so the probability of the event that those who vote for A make up some given coalition C is $(\prod_{i \in C} p_i)(\prod_{i \notin C}(1 - p_i))$. Then A's probability of winning is

$$P_A(p) \;=\; \sum_{C \in \mathcal{M}} \left(\prod_{i \in C} p_i \right) \left(\prod_{i \notin C} (1 - p_i) \right),$$

where $\mathcal{M} = \{C \subseteq N \mid \#C \geq m\}$ is the set of majority coalitions. The mappings $E_A \colon [0,1]^N \rightarrow \Re$ and $P_A \colon [0,1]^N \rightarrow \Re$ are clearly continuous and monotonic,[2] but the similarities end there. Whereas E_A is linear, with gradient $DE_A = (1, \ldots, 1)$, P_A is not even quasi-concave. These mappings for B, E_B and P_B, are defined similarly.

Once expected plurality or probability of winning is specified, a mapping $\pi \colon X \times X \rightarrow [0,1]^N$ from platform pairs to probability vectors defines a two-player, symmetric, constant-sum, simultaneous move game: the players are candidates A and B, each has strategy space X, and A's payoff function is E^{π}_A or P^{π}_A, where

$$\begin{aligned} E^{\pi}_A(x_A, x_B) &= E_A(\pi(x_A, x_B)) \\ P^{\pi}_A(x_A, x_B) &= P_A(\pi(x_A, x_B)), \end{aligned}$$

and B's payoff function is defined similarly. It can be checked that candidate B's payoffs are $n - E_A$ or $1 - P_A$, respectively. When differentiating, we will

[2] If p' is at least as great as p in every component and strictly greater in at least one, then $E_A(p') > E_A(p)$ and $P_A(p') \geq P_A(p)$. If p' is also strictly greater than zero in every component, then $P_A(p') > P_A(p)$).

often want to think of a candidate's payoff function as a function of that candidate's platform alone. We indicate this with a vertical bar, as in

$$DP_A^\pi(x_A|x_B);$$

this notation will indicate the vector of partial derivates of P_A^π with respect to the coordinates of A's platform, evaluated at (x_A, x_B). Similarly, for purposes of differentiation, we treat $\pi(x_A|x_B)$ as a function of x_A only. Let $\pi(x_A, x_B) = (\pi_1(x_A, x_B), \ldots, \pi_n(x_A, x_B))$ denote the values of the π mapping, which for now we leave completely arbitrary.

The platform x_A is a *best response to x_B in neighborhood $G \subseteq \Re^d$ under expected plurality* if $x_A \in G \cap X$ and, for all $x \in G \cap X$,

$$E_A^\pi(x_A, x_B) \quad \geq \quad E_A^\pi(x, x_B).$$

Platform x_A is a *local best response to x_B under expected plurality* if it is a best response in some neighborhood under expected plurality. It is a *best response to x_B under expected plurality* if it is a best response in $G = \Re^d$ under expected plurality. Best reponses for B are defined similarly. A pair (x_A, x_B) of platforms is a *local equilibrium under expected plurality* if x_A is a local best response to x_B under expected plurality, and likewise for x_B. It is an *equilibrium under expected plurality* if x_A is a best response to x_B under expected plurality, and likewise for x_B. The modifier "strict" will indicate that the above inequality actually holds strictly for $x \neq x_A$. We make the same definitions for the probability of winning game by substituting "P_A" for "E_A," and we indicate this using "under probability of winning." Under either objective function, the modifier "symmetric" will indicate that the candidates adopt the same platform, and "interior" will indicate that the candidates' platforms are interior to X.

The idea of a best response can be depicted graphically in the space $[0, 1]^n$ of probability vectors. Let $\mathcal{P}_A(x_B) = \{\pi(x, x_B) \mid x \in X\}$ be the set of probability vectors A can achieve by different choices of platform when B's platform is fixed at x_B. This set is depicted in Figure 1 (drawn as though $n = 2$), assuming symmetry among the candidates, i.e., $\pi(x_B, x_B) = (1/2, \ldots, 1/2)$. The indifference "curves" for expected plurality are just straight lines, and a best response for A (corresponding to (p_1, p_2) in Figure 1) is the platform with the probability vector on the highest achievable indifference curve. Indifference curves for probability of winning are not so nicely behaved, but the idea is the same: the candidate seeks the feasible probability vector on the highest indifference curve.

[Figure 1 about here.]

This graphic intuition suggests an alternative notion of locality for best responses, that used by Patty (1999). We say x_A is a *best response to* x_B *in* p*-neighborhood* $G \subseteq \Re^n$ *under expected plurality* if $\pi(x_A, x_B) \in G$ and, for all $p \in G \cap \mathcal{P}_A(x_B)$,

$$E_A(\pi(x_A, x_B)) \quad \geq \quad E_A(p).$$

We could then define p-local best responses and p-local equilibria paralleling the above definitions. The concepts of local best response and p-local best response are not generally equivalent, but, assuming π is continuous, it is fairly clear that every p-local best response to x_B is a local best response: given p-neighborhood G for which the above inequality holds, the neighborhood $\{x \in X \mid \pi(x, x_B) \in G\}$ of x_A will suffice. It follows, of course, that p-local equilibria must be local equilibria. The other direction need not hold, even when π is continuous. In the model of the next section, we can show that local equilibria under expected plurality must be p-local equilibria under expected plurality, but, since we have no such result for probability of winning, we maintain the original definition of locality.

Our first result establishes one direction of equilibrium equivalence immediately: under concavity and symmetry conditions, a symmetric equilibrium (or even local equilibrium) under probability of winning must be an equilibrium under expected plurality. In Sections 4 and 5, we will give a model of probabilistic voting (the π mapping) under which the conditions of Theorem 1 hold, and we will give some justification for focusing on symmetric equilibria under probability of winning.

Theorem 1 *Assume that* π *is differentiable, that* X *is convex, that* $\pi(\cdot, x^*)$ *is concave and* $\pi(x^*, \cdot)$ *is convex, and that* $\pi(x^*, x^*) = (1/2, \ldots, 1/2)$*. If* (x^*, x^*) *is an interior local equilibrium under probability of winning, then* (x^*, x^*) *is an equilibrium under expected plurality.*

To prove the theorem, we consider A's best response problem. Since x^* is an interior local best response for A under probability of winning, the necessary first order condition must be met:

$$DP_A^\pi(x^*|x^*) \;=\; 0.$$

Using the chain rule, this is

$$\sum_{i \in N} \frac{\partial P_A}{\partial p_i}(1/2, \ldots, 1/2) \frac{\partial \pi_i}{\partial x_j}(x^*|x^*) \;=\; 0,$$

$j = 1, \ldots, d$. Note that

$$\frac{\partial P_A}{\partial p_i}(1/2, \ldots, 1/2) \;=\; \left(\frac{1}{2}\right)^{n-1} \sum_{k=m}^{n} \left[\binom{n-1}{k-1} - \binom{n-1}{k} \right]$$

for all i, where the righthand side is proportional to the number of majorities a voter i belongs to minus the number i does not belong to. Clearly, this is positive and the same for all i. Thus, the first order condition reduces to

$$\sum_{i \in N} \frac{\partial \pi_i}{\partial x_j}(x^*|x^*) \;=\; 0,$$

$j = 1, \ldots, d$, or equivalently,

$$DE_A^\pi(x^*|x^*) \;=\; 0.$$

Since E_A is concave and $\pi(\cdot, x^*)$ is concave, the composition $E_A^\pi(\cdot|x^*)$ is concave as well. Then $DE_A^\pi(x^*|x^*) = 0$ implies that x^* maximizes $E_A^\pi(\cdot|x^*)$. Thus, x^* is a best response for A. A symmetric argument establishes the same for B and completes the proof.

The reader can check that, if we make the stronger assumption that (x^*, x^*) is a p-local equilibrium under probability of winning in Theorem 1, we can replace the concavity-convexity assumption on π with the assumption that $\mathcal{P}_A(x_B)$ and $\mathcal{P}_B(x_A)$ are convex.

3 Discussion

In the remainder of the paper, we will be concerned mostly with the direction of equivalence not covered in Theorem 1, namely, when equilibria under expected plurality are equilibria under probability of winning. Theorem 1, which uses fairly weak assumptions, relies heavily on the concavity of E_A. In considering the other direction, we cannot rely on any such structure in P_A. To anticipate some of the technical problems that can arise going from equilibria under expected plurality to equilibria under probability of winning, suppose $n = 3$ and consider the expected plurality indifference curve through the probability vector $\overline{p} = (1/2, 1/2, 1/2)$. Now consider the vector $p_\epsilon = (1/2 - \epsilon, (1+\epsilon)/2, (1+\epsilon)/2)$, also on that indifference curve. Calculating the probability of winning, we have

$$\begin{aligned} P_A(p_\epsilon) &= 2\left(\frac{1-2\epsilon}{2}\right)\left(\frac{1+\epsilon}{2}\right)\left(\frac{1-\epsilon}{2}\right) \\ &\quad + \left(\frac{1+2\epsilon}{2}\right)\left(\frac{1+\epsilon}{2}\right)^2 \\ &\quad + \left(\frac{1-2\epsilon}{2}\right)\left(\frac{1+\epsilon}{2}\right)^2 \\ &= \frac{1+\epsilon^3}{2}, \end{aligned}$$

which is greater than $P_A(\overline{p}) = 1/2$.

This means that the probability of winning indifference curve through $\overline{p}$ dips below the expected plurality indifference curve, as in Figure 2 (drawn as though $n = 2$), raising the possibility that a deviation from $\overline{p}$ may be profitable under probability of winning but not under expected plurality. Indeed, this possibility is realized if all probability vectors on the expected plurality indifference curve through $\overline{p}$ are feasible, i.e., $\mathcal{P}_A(x_B)$ contains all vectors of probabilities with components summing to $n/2$. And we would expect it to occur when the feasible probability vector frontier is curved but "close enough" to flat at $\overline{p}$. As a consequence, equilibria under expected plurality cannot generally correspond to equilibria under probability of winning — and since p_ϵ can be taken to be arbitrarily close to $\overline{p}$, they cannot generally correspond even to p-local (or local) equilibria under probability of winning.

[Figure 2 about here.]

Let's briefly consider the way in which local equilibria under expected plurality might break down under probability of winning, supposing we have a symmetric, local equilibrium under expected plurality. Suppose, as in Theorem 1, that this equilibrium generates the vector $(1/2, \ldots, 1/2)$ of probabilities, lying on some expected plurality indifference curve, say I. And suppose each neighborhood of candidate A's platform contains a deviation that increases A's probability of winning, generating a sequence $\{p_k\}$ of probability vectors, each giving A a probability of winning greater than one half, converging to $(1/2, \ldots, 1/2)$. Because $DP_A(1/2, \ldots, 1/2)$ is a positive scalar multiple of $DE_A = (1, \ldots, 1)$, as shown in the proof of Theorem 1, the sequence $\{p_k\}$ cannot approach $(1/2, \ldots, 1/2)$ in a straight line from below the indifference curve I — this is essentially shown in the proof of Patty's (1999b) Theorem 4. Such a sequence, if any, would have to approach $(1/2, \ldots, 1/2)$ either on I or from below I along a curve, the possibility of the latter hinging on the flatness of the feasible probability vector frontier. We will see these ideas developed in greater detail in Sections 6 through 8 in a specific model of probabilistic voting.

4 The Additive Bias Model

Now assume that each voter i has a utility function $u_i \colon X \to \Re$ and a utility bias β_i in favor of candidate B, in the sense that i votes for A if and only if the utility of A's platform exceeds that of B's platform by at least β_i. That is, i votes for A if and only if $u_i(x_A) \geq u_i(x_B) + \beta_i$. Assume that X is convex and that each u_i is differentiable and concave. (At times we may assume higher orders of differentiability.) Letting $U = \sum_{i \in N} u_i$, also assume that U is strictly concave.[3] Let $\mathcal{U}_i = \{u_i(x) \mid x \in X\}$ denote the set of possible

[3]This is called "aggregate strict concavity" by Banks and Duggan (1999). See the discussion there regarding the generality gained by weakening the assumption that each u_i is strictly concave.

utilities for i, and let $\overline{u}$ and $\underline{u}$ be finite bounds such that

$$\overline{u} \geq \sup \bigcup_{i \in N} \mathcal{U}_i - \mathcal{U}_i \quad \text{and} \quad \underline{u} \leq \inf \bigcup_{i \in N} \mathcal{U}_i - \mathcal{U}_i.$$

We view the profile $(\beta_1, \ldots, \beta_n)$ of biases as a random variable, and we assume the biases of the voters are distributed independently. Implicitly, then, we assume the support of the biases is wide enough to encompass all possible utility differences of all possible voters. In other words, it is possible that each voter i's bias is so great that the platforms of the candidates would not matter to the voter. For simplicity, we assume each β_i is distributed uniformly on $[\underline{u}, \overline{u}]$.

Given platforms (x_A, x_B), the probability that i votes for candidate A is the probability that $\beta_i \leq u_i(x_A) - u_i(x_B)$, i.e.,

$$p_i = \frac{u_i(x_A) - u_i(x_B) - \underline{u}}{\overline{u} - \underline{u}}.$$

Now add the assumption that $\underline{u} = -\overline{u}$, which means that

$$p_i = \frac{1}{2} + \frac{u_i(x_A) - u_i(x_B)}{\overline{u} - \underline{u}}.$$

In particular, $p_i = 1/2$ when $x_A = x_B$, so the symmetry condition of Theorem 1 is satisfied. Finally, we scale individual utility functions and our bounds so that $\overline{u} - \underline{u} = 1$, i.e., $\overline{u} = 1/2$ and $\underline{u} = -1/2$. Our final expression for the probability that i votes for candidate A is then

$$p_i = \frac{1}{2} + u_i(x_A) - u_i(x_B).$$

We use this to define the mapping π^{ab} associating platform pairs to probability vectors as follows:

$$\pi_i^{ab}(x_A, x_B) = \frac{1}{2} + u_i(x_A) - u_i(x_B),$$

$i = 1, \ldots, n$. The mapping is clearly differentiable, with $D\pi_i^{ab}(x_A|x_B) = Du_i(x_A)$, and satisfies the concavity-convexity condition of Theorem 1. This gives us a particular model, the "additive bias" model, generating the probabilistic voting of Section 2. We will write E^{ab}_A or P^{ab}_A for A's payoff function in this model and E^{ab}_B or P^{ab}_B for B's.

In this model, we can view A's set of feasible probability vectors, $\mathcal{P}_A(x_B)$, in terms of the set $\mathcal{U} = \{(u_1(x), \ldots, u_n(x)) \mid x \in X\}$ of feasible utility imputations. To see this, take any x_B, and note that $\mathcal{P}_A(x_B)$ consists of profiles

$$(u_1(x) - u_1(x_B) + 1/2, \ldots, u_n(x) - u_n(x_B) + 1/2)\,,$$

where x ranges over X. Writing $u(x)$ for utility imputation $(u_1(x), \ldots, u_n(x))$, that is

$$\begin{aligned} \mathcal{P}_A(x_B) &= \{u(x) - u(x_B) + (1/2, \ldots, 1/2) \mid x \in X\} \\ &= \mathcal{U} - u(x_B) + (1/2, \ldots, 1/2), \end{aligned}$$

which is just a translation of $\mathcal{U}$. Given the set $\mathcal{U}$, as in Figure 3 (drawn as though $n = 2$ using the dashed axes), we can then read the set $\mathcal{U} - u(x_B)$ by then translating the origin to $u(x_B)$ (the dotted axes); and we can read the set $\mathcal{P}_A(x_B)$ by translating the origin to $(-1/2, \ldots, -1/2)$ (the solid axes). Note that, as long as x_B is Pareto optimal, $u(x_B)$ will be on the northeast frontier of $\mathcal{U}$ (and therefore of $\mathcal{P}_A(x_B)$). The point $u(x_B)$ will always correspond to the point $(1/2, \ldots, 1/2)$ in $\mathcal{P}_A(x_B)$.

[Figure 3 about here.]

Equilibria under expected plurality in the additive bias model are well-understood. Theorem 8 of Banks and Duggan (1999), for example, establishes that all interior local equilibria under expected plurality are symmetric. Their Corollary 3 establishes that, in any interior, symmetric local equilibrium, the candidates adopt the utilitarian optimum,

$$\overline{x} \;=\; \arg\max_{x \in X} U(x), \tag{1}$$

where U can have at most one maximizer under our assumptions. From our differentiability and concavity assumptions, this platform is a strict maximizer and, when interior to X, is characterized by the first order condition,

$$DU(\overline{x}) \;=\; 0.$$

In Sections 6 through 8, our goal will therefore be to understand when $(\overline{x}, \overline{x})$ is a local equilibrium under probability of winning, i.e., when $x_A = \overline{x}$ is a local best response to $x_B = \overline{x}$.

5 Policy Coincidence

Before proceeding with the analysis of equilibrium equivalence, however, we give some results on probability of winning equilibria in the additive bias model to justify our focus on symmetric equilibria in Theorem 1. Calvert (1985) also proves that all equilibria under probability of winning are symmetric, but, while his model of probabilistic voting is general, he assumes the existence of an "estimated median" (an equilibrium satisfying certain properties). The next result states that, if an interior equilibrium under probability of winning exists, then the candidates must choose the same platform, and that platform must be the utilitarian optimum.

Theorem 2 *In the additive bias model, assume that X is convex, that each u_i is differentiable and concave, that U is strictly concave, and that the β_i are independently and uniformly distributed on $[\underline{u}, \overline{u}] = [-1/2, 1/2]$. If (x_A^*, x_B^*) is an interior equilibrium under probability of winning, then $x_A^* = x_B^* = \overline{x}$.*

The proof of this result follows directly from Theorem 1 and the results of Banks and Duggan (1999). To see this, suppose (x_A^*, x_B^*) is a local equilibrium under probability of winning, and denote $x' = x_A^*$ and $x'' = x_B^*$. Since the campaign competition game is symmetric, (x'', x') is also an equilibrium. And since it is constant sum, equilibrium strategies are interchangeable: (x', x') and (x'', x'') are interior equilibria under probability of winning. Then Theorem 1 implies they are equilibria under expected plurality, and then Theorem 6 of Banks and Duggan implies $x' = x'' = \overline{x}$. The consequence of this result, that the candidates in equilibrium must adopt the same policies, is referred to by Banks and Duggan as "policy coincidence." In contrast to their Theorem 8, however, Theorem 2 applies only to global — not to mere local — equilibria. The next result extends policy coincidence to local equilibria using the assumption that each u_i is *quadratic,* i.e., there is some $x^i \in X$ such that $u_i(x) = -||x^i - x||^2$.

Theorem 3 *In the additive bias model, assume that each u_i is quadratic and that the β_i are independently uniformly distributed on $[\underline{u}, \overline{u}] = [-1/2, 1/2]$. If (x_A^*, x_B^*) is an interior local equilibrium under probability of winning, then $x_A^* = x_B^* = \overline{x}$.*

The proof of this result rests on a closer look at the candidates' first order conditions and on some elementary facts about quadratic utilities. Fixing the candidates at interior platforms (x_A, x_B) and taking the partial derivative of P_A^{ab} with respect to the jth coordinate of x_A, we have

$$\frac{\partial P_A^{ab}}{\partial x_j}(x_A|x_B) \;=\; \sum_{i \in N} \frac{\partial P_A}{\partial p_i}(p) \frac{\partial \pi_i^{ab}}{\partial x_j}(x_A|x_B),$$

where $p = \pi^{ab}(x_A, x_B)$. Letting

$$\begin{aligned} \mathcal{M}_i^+ &= \{C \in \mathcal{M} \mid i \in C\} \\ \mathcal{M}_i^- &= \{C \in \mathcal{M} \mid i \notin C\} \end{aligned}$$

denote the collections of majority coalitions containing and not containing voter i, respectively, note that

$$\begin{aligned} \frac{\partial P_A}{\partial p_i}(p) \;=\; & \sum_{C \in \mathcal{M}_i^+} \left(\prod_{j \in C, j \neq i} p_j \right) \left(\prod_{j \notin C} (1 - p_j) \right) \\ & - \sum_{C \in \mathcal{M}_i^-} \left(\prod_{j \in C} p_j \right) \left(\prod_{j \notin C, j \neq i} (1 - p_j) \right). \end{aligned}$$

Furthermore, each term corresponding to $C \in \mathcal{M}_i^-$ cancels with the term corresponding to $C \cup \{i\} \in \mathcal{M}_i^+$. Thus, letting

$$\mathcal{H}_i^- \;=\; \{C \subseteq N \mid i \notin C, \#C = n/2\}$$

denote the collection of coalitions containing exactly half of the voters but not containing i, we have

$$\frac{\partial P_A}{\partial p_i}(p) \;=\; \sum_{C \in \mathcal{H}_i^-} \left(\prod_{j \in C} p_j \right) \left(\prod_{j \notin C, j \neq i} (1 - p_j) \right),$$

which is just the probability that the voters other than i split between candidates A and B. To simplify notation, denote this by $\sigma_i(p)$. Thus, we have established that

$$\frac{\partial P_A^{ab}}{\partial x_j}(x_A|x_B) = \sum_{i\in N} \sigma_i(\pi^{ab}(x_A|x_B))\frac{\partial u_i}{\partial x_j}(x_A).$$

Now let (x_A^*, x_B^*) be an interior local equilibrium under probability of winning, and, for simplicity, let $\sigma_i^* = \sigma_i(\pi^{ab}(x_A^*, x_B^*))$. Then the first order necessary condition must hold for candidate A:

$$\frac{\partial P_A^{ab}}{\partial x_j}(x_A^*|x_B^*) = \sum_{i\in N} \sigma_i^* \frac{\partial u_i}{\partial x_j}(x_A^*) = 0, \qquad (2)$$

$j = 1, \ldots, d$. It is well known that $Du_i(x) = 2(x^i - x)$ when u_i is quadratic. Setting $\hat{t} = x_B^* - x_A^*$, the derivative of u_i in direction $\hat{t}$, evaluated at x_A^*, is then

$$D_{\hat{t}}u_i(x_A^*) = 2x^i \cdot (x_B^* - x_A^*) - 2x_A^* \cdot x_B^* + 2x_A^* \cdot x_A^*.$$

Furthermore, it is easily checked that

$$u_i(x_B^*) - u_i(x_A^*) = 2x^i \cdot (x_B^* - x_A^*) + x_A^* \cdot x_A^* - x_B^* \cdot x_B^*.$$

Therefore,

$$\begin{aligned} D_{\hat{t}}P(x_A^*|x_B^*) &= \sum_{i\in N} \sigma_i^* D_{\hat{t}}u_i(x_A^*) \\ &= \sum_{i\in N} \sigma_i^*(u_i(x_B^*) - u_i(x_A^*) + x_A^* \cdot x_A^* - 2x_A^* \cdot x_B^* + x_B^* \cdot x_B^*) \\ &= 0. \end{aligned}$$

Since

$$x_A^* \cdot x_A^* - 2x_A^* \cdot x_B^* + x_B^* \cdot x_B^* = ||x_A^* - x_B^*||^2,$$

this is equivalent to

$$\sum_{i\in N} \sigma_i^*(u_i(x_B^*) - u_i(x_A^*)) = -||x_A^* - x_B^*||^2 \sum_{i\in N} \sigma_i^*. \qquad (3)$$

A symmetric analysis of candidate B's first order condition yields

$$\sum_{i\in N}\sigma_i^*(u_i(x_A^*)-u_i(x_B^*)) \;=\; -||x_A^*-x_B^*||^2\sum_{i\in N}\sigma_i^*. \qquad (4)$$

Since the candidate competition game is symmetric, it follows that the candidates' payoffs are one half in equilibrium, and therefore that at least half of the voters vote for candidate A with positive probability. Therefore, $\sum_{i\in N}\sigma_i^* > 0$. Then (3) and (4) imply $||x_A^*-x_B^*||^2 = 0$, which implies $x_A^* = x_B^*$. Finally, $x_A^* = x_B^* = \overline{x}$ follows from Theorem 1 and from Theorem 6 of Banks and Duggan (1999).

6 Local Equilibria

Our goal is to understand when $\overline{x}$ is a best response for candidate A to $\overline{x}$, i.e., when $\overline{x}$ solves

$$\max_{x\in X} P_A^{ab}(x|\overline{x}).$$

In the additive bias model with interior $\overline{x}$, the necessary first order condition (see (2)) reduces to

$$DP_A^{ab}(\overline{x}|\overline{x}) \;=\; \left(\frac{1}{2}\right)^{n-1}\sum_{k=m}^{n}\left[\binom{n-1}{k-1}-\binom{n-1}{k}\right]DU(\overline{x}) \;=\; 0$$

at $\overline{x}$. Then $DU(\overline{x}) = 0$ implies that the first order condition is satisfied. As we will see, however, much hinges on the second order condition for a maximum. Taking the cross partial of P_A^{ab} at $\overline{x}$, we have

$$\frac{\partial^2 P_A^{ab}}{\partial x_j\partial x_k}(x_A|x_B) \;=\; \sum_{i\in N}\left[\frac{\partial P_A}{\partial p_i}(p)\frac{\partial^2\pi_i^{ab}}{\partial x_j\partial x_k}(x_A|x_B)+A_i\frac{\partial \pi_i^{ab}}{\partial x_j}(x_A|x_B)\right],$$

where $p = \pi^{ab}(x_A, x_B)$ and the A_i are terms that depend on i. Evaluating this cross partial at $(\overline{x},\overline{x})$, we have

$$\frac{\partial^2 P_A^{ab}}{\partial x_j\partial x_k}(\overline{x}|\overline{x}) \;=\; \left(\frac{1}{2}\right)^{n-1}\left[\binom{n-1}{m-1}-\binom{n-1}{m}\right]\frac{\partial^2 U}{\partial x_j\partial x_k}(\overline{x}),$$

where the A_i terms are independent of i and drop out because $DU(\overline{x}) = 0$. Therefore, $D^2 P_A^{ab}(\overline{x}|\overline{x})$ is a positive scalar multiple of $D^2 U(\overline{x})$, and so the definiteness of one matrix determines the definiteness of the other.

This gives us a simple sufficient condition under which $\overline{x}$ is a local maximizer of candidate A's (and therefore B's) best response problem, i.e., $(\overline{x}, \overline{x})$ is a local equilibrium under probability of winning. In Section 9, we consider a strengthening sufficient for global equilibrium equivalence.

Theorem 4 *In the additive bias model, assume that X is convex, that each u_i is twice differentiable and concave, that $\overline{x}$ is interior, and that the β_i are independently uniformly distributed on $[\underline{u}, \overline{u}] = [-1/2, 1/2]$. If $D^2 u_i(\overline{x})$ is negative definite for some i, then $(\overline{x}, \overline{x})$ is a local equilibrium under probability of winning.*

Thus, a slight strengthening of our concavity assumptions delivers equivalence of local equilibria: we initially assumed that each u_i was concave and that U was strictly so, and we need only add negative definiteness of some $D^2 u_i(\overline{x})$ (and therefore of $D^2 U(\overline{x})$). It follows that, when $D^2 U(\overline{x})$ is negative definite, the frontier of the set $\mathcal{P}_A(\overline{x})$ of probability vectors feasible for A cannot be "too flat" at $(1/2, \ldots, 1/2)$, and the problem alluded to at the end of Section 3 cannot arise. Even assuming each u_i is *strictly* concave, however, we are ensured only that $D^2 U(\overline{x})$ is negative definite *almost everywhere.* A possibility left open in Theorem 4 is that, though the voters all have strictly concave utility functions, $D^2 U(\overline{x})$ is only negative *semi*-definite at $\overline{x}$, leading to a breakdown of the local equilibrium equivalence result. We will see that, perhaps surprisingly, this can be the case.

7 Three Voters

In this section, we consider the $n = 3$ case of the additive bias model in detail. Fixing candidate B at $\overline{x}$, suppose candidate A deviates from $\overline{x}$ to some platform x. For each i, let $\Delta_i = u_i(x) - u_i(\overline{x})$ denote the increase in the probability that i votes for A as a result of the deviation. The probability that i votes for A is then $1/2 + \Delta_i$ (and the probability that i votes for B is

$1/2 - \Delta_i$). Thus, candidate A's probability of winning after deviating is

$$\begin{aligned} P_A^{ab}(x|\overline{x}) &= (1/2 + \Delta_1)(1/2 + \Delta_2)(1/2 - \Delta_3) \\ &\quad +(1/2 + \Delta_1)(1/2 - \Delta_2)(1/2 + \Delta_3) \\ &\quad +(1/2 - \Delta_1)(1/2 + \Delta_2)(1/2 + \Delta_3) \\ &\quad +(1/2 + \Delta_1)(1/2 + \Delta_2)(1/2 + \Delta_3) \\ &= 1/2 + (\Delta_1 + \Delta_2 + \Delta_3)/2 - 2(\Delta_1\Delta_2\Delta_3). \end{aligned} \tag{5}$$

We want to understand when an arbitrarily small deviation will increase A's probability of winning, i.e., when x arbitrarily close to $\overline{x}$ will satisfy

$$(\Delta_1 + \Delta_2 + \Delta_3)/2 - 2(\Delta_1\Delta_2\Delta_3) \quad > \quad 0.$$

Because $\overline{x}$ is a strict maximizer of (1), the first term above will always be negative. A necessary condition for a profitable deviation is, therefore, that the second term in parentheses be negative, i.e., that Δ_i be positive for two voters and negative for one. Therefore, if $\overline{x}$ is a core point (no platform x is strictly preferred by a majority of voters), then $(\overline{x}, \overline{x})$ is a local equilibrium under probability of winning.

In fact, we will consider the somewhat narrower question of when there exist arbitrarily small profitable deviations in a given direction $\hat{t}$. If candidate A moves a distance of ϵ in direction $\hat{t}$, the change in utility for voter i is $\Delta_i^\epsilon = u_i(\overline{x} + \epsilon\hat{t}) - u_i(\overline{x})$. Write $(\Delta_1^\epsilon + \Delta_2^\epsilon + \Delta_3^\epsilon)/2 - 2(\Delta_1^\epsilon\Delta_2^\epsilon\Delta_3^\epsilon)$ as

$$\frac{1}{2}\left(\Delta_1^\epsilon + \Delta_2^\epsilon + \Delta_3^\epsilon\right) - 2\epsilon^3\left(\frac{\Delta_1^\epsilon}{\epsilon}\frac{\Delta_2^\epsilon}{\epsilon}\frac{\Delta_3^\epsilon}{\epsilon}\right), \tag{6}$$

and let ϵ go to zero. Note that $\Delta_1^\epsilon + \Delta_2^\epsilon + \Delta_3^\epsilon$ converges to zero, and, since $\overline{x}$ is a strict maximizer in (1), it does so from below. Thus, the quantity in (6) is positive for sufficiently small ϵ if and only if $\Delta_1^\epsilon\Delta_2^\epsilon\Delta_3^\epsilon$ is negative for sufficiently small ϵ and $\Delta_1^\epsilon + \Delta_2^\epsilon + \Delta_3^\epsilon$ goes to zero faster than ϵ^3. The former condition can be simplified by noting it implies, for some ϵ, that $u_i(\hat{x} + \epsilon\hat{t}) > u_i(\overline{x})$ for two voters, the opposite equality for the other. Then concavity of the u_i yields

$$D_{\hat{t}}u_1(\overline{x})D_{\hat{t}}u_2(\overline{x})D_{\hat{t}}u_3(\overline{x}) \quad < \quad 0,$$

a condition we express by writing that "direction $\hat{t}$ is majority-preferred to $\overline{x}$." Conversely, since

$$\lim_{\epsilon \to 0} \left(\frac{\Delta_1^\epsilon}{\epsilon} \frac{\Delta_2^\epsilon}{\epsilon} \frac{\Delta_3^\epsilon}{\epsilon} \right) \quad = \quad D_{\hat{t}} u_1(\overline{x}) D_{\hat{t}} u_2(\overline{x}) D_{\hat{t}} u_3(\overline{x}),$$

this implies that $\Delta_1^\epsilon \Delta_2^\epsilon \Delta_3^\epsilon$ is negative for sufficiently small ϵ

To be more formal about rates of convergence, let $\{a_k\}$ and $\{b_k\}$ be sequences decreasing to zero and let $\{c_k\}$ converge to c. The sign of $a_k - b_k c_k$ is the same as the sign of $(a_k/b_k) - c_k$. Since $c_k \to c$, it is positive for high enough k if $\lim(a_k/b_k) > c$ and only if $\lim(a_k/b_k) \geq c$. Noting that $\Delta_1^\epsilon + \Delta_2^\epsilon + \Delta_3^\epsilon = U(\overline{x} + \epsilon \hat{t}) - U(\overline{x})$, this means that the quantity in (6) is positive for sufficiently small ϵ if

$$\lim_{\epsilon \to 0} \frac{U(\overline{x} + \epsilon \hat{t}) - U(\overline{x})}{\epsilon^3} \quad > \quad 4 D_{\hat{t}} u_1(\overline{x}) D_{\hat{t}} u_2(\overline{x}) D_{\hat{t}} u_3(\overline{x}),$$

where the limit on the left is to be determined. Note that it must be non-positive, by definition of $\overline{x}$, so the above strict inequality implies that $\hat{t}$ is majority-preferred to $\overline{x}$. Applying L'Hôpital's rule (cf. Apostol, 1974, Exercise 5.28), the limit on the left is given by

$$\frac{D_{\hat{t}} U(\overline{x})}{\lim_{\epsilon \to 0} 3\epsilon^2},$$

which is still indeterminate, since $DU(\overline{x}) = 0$. Applying L'Hôpital's rule again, the limit is

$$\frac{D_{\hat{t}}^2 U(\overline{x})}{\lim_{\epsilon \to 0} 6\epsilon}.$$

This is $-\infty$ if $D_{\hat{t}}^2 U(\overline{x}) < 0$, in which case the quantity in (6) is negative for small enough ϵ, and it is indeterminate if $D_{\hat{t}}^2 U(\overline{x}) = 0$. In that case, we apply L'Hôpital's rule again to get the limit

$$D_{\hat{t}}^3 U(\overline{x})/6.$$

Thus, we find that the quantity in (6) is positive for sufficiently small ϵ if $D_{\hat{t}}^2 U(\overline{x}) = 0$ and

$$D_{\hat{t}}^3 U(\overline{x}) \quad > \quad 24 D_{\hat{t}} u_1(\overline{x}) D_{\hat{t}} u_2(\overline{x}) D_{\hat{t}} u_3(\overline{x}).$$

Conversely, (6) is positive for sufficiently small ϵ only if $\hat{t}$ is majority-preferred to $\overline{x}$, $D^2_{\hat{t}}U(\overline{x}) = 0$, and the above inequality holds weakly.

The results of this discussion are summarized in the next theorem.

Theorem 5 *In the additive bias model, assume that $n = 3$, that X is convex, that each u_i is three-times differentiable and concave, that $\overline{x}$ exists and is interior to X, and that the β_i are independently uniformly distributed on $[\underline{u}, \overline{u}] = [-1/2, 1/2]$. If*

$$D^2_{\hat{t}}U(\overline{x}) = 0 \quad \text{and} \quad D^3_{\hat{t}}U(\overline{x}) > 24 D_{\hat{t}}u_1(\overline{x}) D_{\hat{t}}u_2(\overline{x}) D_{\hat{t}}u_3(\overline{x}), \tag{7}$$

then $P^{ab}_A(\overline{x}+\epsilon\hat{t}|\overline{x}) > 1/2$ for sufficiently small ϵ. Conversely, if $P^{ab}_A(\overline{x}+\epsilon\hat{t}|\overline{x}) > 1/2$ for sufficiently small ϵ, then $\hat{t}$ is majority-preferred to $\overline{x}$ and (7) holds with a weak inequality.

Theorem 5 gives conditions under which $(\overline{x}, \overline{x})$ is not a local equilibrium (and therefore not an equilibrium) under probability of winning. Combining this result with Theorem 2, we have the following corollary, which gives general conditions for non-existence of equilibrium under probability of winning. Note that the probability of winning objective function is continuous and the strategy spaces of the candidates can be taken to be compact as well as convex — thus, the standard results on equilibrium existence in non-cooperative games would apply if the probability of winning objective function were quasi-concave. This pinpoints the cause of non-existence. A similar corollary could be stated for local equilibria using Theorem 3 instead.

Corollary 1 *In the additive bias model, assume that $n = 3$, that X is convex, that each u_i is three-times differentiable and concave, that U is strictly concave, that $\overline{x}$ exists and is interior to X, and that the β_i are independently uniformly distributed on $[\underline{u}, \overline{u}] = [-1/2, 1/2]$. If*

$$D^2_{\hat{t}}U(\overline{x}) = 0 \quad \text{and} \quad D^3_{\hat{t}}U(\overline{x}) > 24 D_{\hat{t}}u_1(\overline{x}) D_{\hat{t}}u_2(\overline{x}) D_{\hat{t}}u_3(\overline{x}),$$

then there is no interior equilibrium under probability of winning.

Let's consider a simple environment in which the conditions of Theorem 5 and Corollary 1 hold. Let $d = 1$, and define voter utility functions as follows:

$$\begin{aligned} u_1(x) &= -2x - 4(x - 1/2)^4 \\ u_2(x) &= x - 2(x - 1/2)^4 \\ u_3(x) &= u_2(x). \end{aligned}$$

It is straightforward to verify that 1's, 2's, and 3's ideal points are 0, 1, and 1, respectively, and that $\overline{x} = 1/2$. Thus, $(1/2, 1/2)$ is the unique equilibrium under expected plurality. Clearly, however, direction $\hat{t} = 1$ is majority-preferred to $\overline{x}$: $u_2'(\overline{x}) = u_3'(\overline{x}) > 0$ and $u_1'(\overline{x}) < 0$. Lastly, note that

$$\begin{aligned} D^2_{\hat{t}} U(\overline{x}) &= u_1''(\overline{x}) + u_2''(\overline{x}) + u_3''(\overline{x}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} D^3_{\hat{t}} U(\overline{x}) &= u_1'''(\overline{x}) + u_2'''(\overline{x}) + u_3'''(\overline{x}) \\ &= 0, \end{aligned}$$

fulfilling condition (7). Thus, arbitrarily small increases in A's platform will produce a probability of winning greater than one half, and we conclude that $(1/2, 1/2)$ is not a local equilibrium under probability of winning.

Taking a sequence $\{x_k\}$ with $x_k \downarrow \overline{x}$ in the above example, candidate A's probability of winning will be greater than one half for high enough k. Letting $\{p_k\}$ be the corresponding sequence of probability vectors, defined by $p_k = \pi^{ab}(x_k, \overline{x})$, this sequence of vectors converges to $(1/2, \ldots, 1/2)$; it is on or below the expected plurality indifference curve through $(1/2, \ldots, 1/2)$ but (eventually) above the probability of winning indifference curve. Thus, we are exploiting the "dip" mentioned at the end of Section 3. And, as mentioned there, this sequence of probability vectors must move toward $(1/2, \ldots, 1/2)$ along a curve. (Recall Figure 2.) Though the platforms $\{x_k\}$ move toward $\overline{x}$ in a line, the corresponding probability vectors, if they are to generate a probability of winning greater than one half, cannot lie on a line.

Several more remarks about Theorem 5 are in order. First, an immediate consequence of the theorem is that, if $D^2_{\hat{t}} U(\overline{x}) = D^3_{\hat{t}} U(\overline{x}) = 0$ and no voter's derivative at $\overline{x}$ in direction $\hat{t}$ is zero, i.e.,

$$D_{\hat{t}}u_1(\overline{x})D_{\hat{t}}u_2(\overline{x})D_{\hat{t}}u_3(\overline{x}) \quad \neq \quad 0,$$

then $(\overline{x}, \overline{x})$ is not a local equilibrium: if the latter product is negative, then moving in direction $\hat{t}$ will be profitable; if it is positive, then use direction $-\hat{t}$. Note also that $D^2_{\hat{t}}U(\overline{x}) = 0$ is inconsistent with negative definiteness of $D^2U(\overline{x})$, as required by Theorem 4: if $D^2U(\overline{x})$ were negative definite, then $(\overline{x}, \overline{x})$ would be a local equilibrium under probability of winning, and the deviations found in Theorem 5 could not be profitable.

Second, condition (7) is not only sufficient for $(\overline{x}, \overline{x})$ to not be a local equilibrium under probability of winning, but it ensures the existence of arbitrarily small profitable deviations of a very specific sort: these deviations from $\overline{x}$ are all in one direction, namely $\hat{t}$. Conversely, the condition is necessary only for the existence of arbitrarily small profitable deviations *in a given direction.* We have not ruled out the possibility that arbitrarily small profitable deviations can be found by moving toward $\overline{x}$ in an arc, though not by moving toward $\overline{x}$ on any fixed line. Suppose, however, that $d = 1$ and that there exists a sequence $\{x_k\}$ of profitable deviations from $\overline{x}$ for candidate A such that $x_k \rightarrow \overline{x}$. Since $d = 1$, there is either a subsequence with $x_{k_l} \uparrow \overline{x}$ or a subsequence with $x_{k_l} \downarrow \overline{x}$. In either case, we can approach $\overline{x}$ by arbitrarily small deviations in one direction, i.e., there exists $\hat{t}$ such that $P(\overline{x}+\epsilon\hat{t}) > 1/2$ for small enough ϵ, so (7) must hold.

Third, our results demonstrate a type of sensitivity that may seem unusual. Assuming $d = 1$ and $u_1'''(\overline{x}) + u_2'''(\overline{x}) + u_3'''(\overline{x}) = 0$ for purposes of illustration, Theorem 4 entails that $(\overline{x}, \overline{x})$ will be a local equilibrium under probability of winning as long as $u_1''(\overline{x})+u_2''(\overline{x})+u_3''(\overline{x})$ is negative, no matter how close to zero. But if the sum of second derivatives equals zero, it is no longer a local equilibrium. Technically, this is a violation of upper hemicontinuity, a property usually demonstrated by equilibrium correspondences. What this illustrates is that *local* equilibrium correspondences need *not* possess this property. If we consider a sequence of utility functions for the voters with the sum of second derivatives converging to zero from below, for each assignment of utility functions there will be some open set G_k containing $\overline{x}$ in which $\overline{x}$ is a best response for the candidates, but $G_k \downarrow \{\overline{x}\}$ — there is no "limiting" open set with this property.

8 Arbitrary Odd Number of Voters

A question remaining from the previous section is whether the fragility of equivalence for local equilibria with $n = 3$ carries over to the model with larger numbers of voters. Another question is whether the existence of a direction $\hat{t}$ majority-preferred to $\overline{x}$ is needed, as it is in the $n = 3$ case, to break that equivalence. We will see that the answers to these questions are "yes, it does," and "no, it is not." As a first step, we develop the insight on which our analysis of the $n = 3$ case was built.

As before, suppose candidate B's platform is $\overline{x}$, and suppose candidate A moves from $\overline{x}$ to some platform x. The change in the probability that voter i votes for A is then $\Delta_i = u_i(x) - u_i(\overline{x})$, and the probability that A wins is

$$P_A^{ab}(x|\overline{x}) \;=\; \sum_{C \in \mathcal{M}} \left(\prod_{i \in C} (1/2 + \Delta_i) \right) \left(\prod_{i \notin C} (1/2 - \Delta_i) \right). \tag{8}$$

Using Δ_C to denote $\prod_{i \in C} \Delta_i$, we want to know the coefficient of an arbitrary $\Delta_{C'}$ when this expression is expanded. Let C' contain exactly c' voters, and take any $C \in \mathcal{M}$ containing c voters. Suppose C contains all but r members of C', i.e., $\#(C' \setminus C) = r$. Thus, $\#(C' \cap C) = c' - r$. The expansion of $\prod_{i \in C}(1/2 + \Delta_i)$ will contain the term

$$(1/2)^{c-(c'-r)} \Delta_{C \cap C'};$$

and the expansion of $\prod_{i \notin C}(1/2 - \Delta_i)$ will contain the term

$$(-1)^r (1/2)^{(n-c)-r} \Delta_{C' \setminus C}.$$

When multiplied, these terms yield $(-1)^r (1/2)^{n-c'} \Delta_{C'}$. The number of majority coalitions containing all but r members of C' is

$$\binom{c'}{r} \sum_{k=\max\{m-c'+r,0\}}^{n-c'} \binom{n-c'}{k},$$

where we use the convention $\binom{n-c'}{0} = 1$. This yields the following lemma.

Lemma 1 *Given coalition $C' \subseteq N$ with c' members, the coefficient of $\Delta_{C'}$ in the expansion of*

$$\sum_{C \in \mathcal{M}} \left(\prod_{i \in C} (1/2 + \Delta_i) \right) \left(\prod_{i \notin C} (1/2 - \Delta_i) \right).$$

is

$$\Gamma_{c'} = \left(\frac{1}{2}\right)^{n-c'} \sum_{r=0}^{c'} (-1)^r \binom{c'}{r} \sum_{k=\max\{m-c'+r,0\}}^{n-c'} \binom{n-c'}{k}.$$

Using the notation of the lemma, the probability that candidate A wins is

$$\Gamma_0 + \Gamma_1 \left(\sum_{i \in N} \Delta_i \right) + \Gamma_3 \left(\sum_{i<j<k} \Delta_i \Delta_j \Delta_k \right) + \sum_{k=4}^{n} \Gamma_k \left(\sum_{\substack{C \subseteq N \\ \#C = k}} \Delta_C \right),$$

where $\Gamma_0 = 1/2$, as in the $n = 3$ case. Note that the coefficient of every one-member coalition is

$$\begin{aligned} \Gamma_1 &= \left(\frac{1}{2}\right)^{n-1} \left[\left(\sum_{k=m-1}^{n-1} \binom{n-1}{k} \right) - \left(\sum_{k=m}^{n-1} \binom{n-1}{k} \right) \right] \\ &= \left(\frac{1}{2}\right)^{n-1} \binom{n-1}{m-1}, \end{aligned}$$

which is greater than zero, as in the $n = 3$ case. The coefficient of every two-member coalition is

$$\left(\frac{1}{2}\right)^{n-2} \left[\left(\sum_{k=m-2}^{n-2} \binom{n-2}{k} \right) - 2 \left(\sum_{k=m-1}^{n-2} \binom{n-2}{k} \right) + \left(\sum_{k=m}^{n-2} \binom{n-2}{k} \right) \right].$$

Using the assumption that n is odd, so that $m = (n+1)/2$, these terms cancel: as in the $n = 3$ case, the coefficient of every two-member coalition is zero. The coefficient of every three-member coalition is

$$\Gamma_3 = \left(\frac{1}{2}\right)^{n-3} \left[\left(\sum_{k=m-3}^{n-3} \binom{n-3}{k} \right) - 3 \left(\sum_{k=m-2}^{n-3} \binom{n-3}{k} \right) \right.$$

$$+3\left(\sum_{k=m-1}^{n-3}\binom{n-3}{k}\right)-\left(\sum_{k=m}^{n-3}\binom{n-3}{k}\right)\Bigg]$$

$$=\binom{n-3}{m-3}-2\binom{n-3}{m-2}+\binom{n-3}{m-1},$$

where we may assume $m \geq 3$, since the $n = 3$ case has been covered. Simplifying and using $m = (n+1)/2$, it is apparent that this expression is negative, as in the $n = 3$ case.

As in Section 7, if candidate A moves a distance of ϵ in direction $\hat{t}$, then the change in utility for voter i is $\Delta_i^\epsilon = u_i(\overline{x} + \epsilon\hat{x}) - u_i(\overline{x})$. We use Δ_C^ϵ to denote $\prod_{i\in C} \Delta_i^\epsilon$. Thus, A's probability of winning is greater than one half when adopting platform $\overline{x} + \epsilon\hat{x}$ if and only if

$$\Gamma_1\left(\sum_{i\in N}\Delta_i^\epsilon\right)+\Gamma_3\left(\sum_{i<j<k}\Delta_i^\epsilon\Delta_j^\epsilon\Delta_k^\epsilon\right)+\sum_{k=4}^{n}\Gamma_k\left(\sum_{\substack{C\subseteq N\\ \#C=k}}\Delta_C^\epsilon\right) \;>\; 0,$$

or equivalently, if and only if

$$\Gamma_1\left(\frac{1}{\epsilon^3}\sum_{i\in N}\Delta_i^\epsilon\right)+\Gamma_3\left(\sum_{i<j<k}\frac{\Delta_i^\epsilon}{\epsilon}\frac{\Delta_j^\epsilon}{\epsilon}\frac{\Delta_k^\epsilon}{\epsilon}\right)+\sum_{k=4}^{n}\Gamma_k\left(\sum_{\substack{C\subseteq N\\ \#C=k}}\frac{\Delta_C^\epsilon}{\epsilon^3}\right) \qquad (9)$$

is positive. Note that the third term in (9) goes to zero: for every coalition C with at least four members, including i, j, and k, say, the term $\Delta_C^\epsilon/\epsilon^3$ goes to

$$D_{\hat{t}}u_i(\overline{x})D_{\hat{t}}u_j(\overline{x})D_{\hat{t}}u_k(\overline{x})\prod_{\substack{h\in C\\ h\neq i,j,k}}\lim_{\epsilon\to 0}\Delta_h^\epsilon \;=\; 0.$$

Thus, following the analysis of the previous section, the sign of (9) is positive for sufficiently small ϵ if $D^2U(\overline{x}) = 0$ and

$$D_{\hat{t}}^3U(\overline{x}) \;>\; -\frac{\Gamma_3}{\Gamma_1}\sum_{i<j<k}D_{\hat{t}}u_i(\overline{x})D_{\hat{t}}u_j(\overline{x})D_{\hat{t}}u_k(\overline{x}).$$

Conversely, (9) is positive for sufficiently small ϵ only if $D^2U(\overline{x}) = 0$ and the above inequality holds weakly.

In Section 7, the condition that $\sum_{i<j<k} D_{\hat{t}}u_i(\overline{x})D_{\hat{t}}u_j(\overline{x})D_{\hat{t}}u_k(\overline{x}) < 0$ was equivalent to $\hat{t}$ being majority-preferred to $\overline{x}$. As we illustrate with an example after Theorem 6, this is no longer the case for general n. The next lemma reduces the summation over triples i, j, k to a sum over i in that condition, simplifying the statement of our results and the computation of examples.

Lemma 2

$$\sum_{i<j<k} D_{\hat{t}}u_i(\overline{x})D_{\hat{t}}u_j(\overline{x})D_{\hat{t}}u_k(\overline{x}) \;=\; \frac{1}{3}\sum_{i\in N}(D_{\hat{t}}u_i(\overline{x}))^3.$$

To prove the lemma, let $\alpha_i = D_{\hat{t}}u_i(\overline{x})$, and write the lefthand side of the above equality as $\sum_{i<j<k}\alpha_i\alpha_j\alpha_k$, where $\sum_{i\in N}\alpha_i = 0$. Then

$$\sum_{i<j<k}\alpha_i\alpha_j\alpha_k \;=\; \frac{1}{3}\sum_{i\in N}\alpha_i\sum_{\substack{j<k\\ j,k\neq i}}\alpha_j\alpha_k. \tag{10}$$

Also,

$$\begin{aligned}
\sum_{\substack{j<k\\ j,k\neq i}}\alpha_j\alpha_k &= \left(\sum_{j<k}\alpha_j\alpha_k\right) - \left(\alpha_i\sum_{j\neq i}\alpha_j\right)\\
&= \frac{1}{2}\left[\left(\sum_{j\in N}\alpha_j\right)^2 - \left(\sum_{j\in N}\alpha_j^2\right)\right] - \alpha_i\left(\sum_{j\neq i}\alpha_j\right)\\
&= -\frac{1}{2}\left(\sum_{j\in N}\alpha_j^2\right) - \alpha_i(-\alpha_i)\\
&= \alpha_i^2 - \frac{1}{2}||\alpha||^2,
\end{aligned}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$. Substituting this into (10), we have

$$\sum_{i<j<k}\alpha_i\alpha_j\alpha_k \;=\; \frac{1}{3}\sum_{i\in N}\alpha_i\left(\alpha_i^2 - \frac{1}{2}||\alpha||^2\right)$$

$$= \frac{1}{3}\left(\sum_{i\in N}\alpha_i^3\right) - \frac{1}{6}||\alpha||^2\left(\sum_{i\in N}\alpha_i\right)$$
$$= \frac{1}{3}\sum_{i\in N}\alpha_i^3,$$

completing the proof of the lemma.

We can now state our result for the case of an arbitrary odd number of voters.

Theorem 6 *In the additive bias model, assume that X is convex, that each u_i is three-times differentiable and concave, that $\overline{x}$ exists and is interior to X, and that the β_i are independently and uniformly distributed on $[\underline{u}, \overline{u}] = [-1/2, 1/2]$. If*

$$D_{\hat{t}}^2 U(\overline{x}) = 0 \quad \text{and} \quad D_{\hat{t}}^3 U(\overline{x}) > -\frac{\Gamma_3}{3\Gamma_1}\sum_{i\in N}(D_{\hat{t}} u_i(\overline{x}))^3, \tag{11}$$

then $P_A^{ab}(\overline{x}+\epsilon\hat{t}|\overline{x}) > 1/2$ for sufficiently small ϵ. Conversely, if $P_A^{ab}(\overline{x}+\epsilon\hat{t}|\overline{x}) > 1/2$ for sufficiently small ϵ, then (11) holds with a weak inequality.

As in Section 7, we have the following corollary on non-existence of equilibrium under probability of winning.

Corollary 2 *In the additive bias model, assume that X is convex, that each u_i is three-times differentiable and concave, that U is strictly concave, that $\overline{x}$ exists and is interior to X, and that the β_i are independently and uniformly distributed on $[\underline{u}, \overline{u}] = [-1/2, 1/2]$. If*

$$D_{\hat{t}}^2 U(\overline{x}) = 0 \quad \text{and} \quad D_{\hat{t}}^3 U(\overline{x}) > -\frac{\Gamma_3}{3\Gamma_1}\sum_{i\in N}(D_{\hat{t}} u_i(\overline{x}))^3,$$

then there is no interior equilibrium under probability of winning.

For an example in which the conditions of Theorem 6 and Corollary 2 hold, we may simply take the example following Theorem 5 and replicate the voters any odd number of times. The comments following that theorem regarding the nature of the deviations found apply here as well: they are of a specific form, namely, they line up in a given direction; this strengthens the sufficiency half of the result and weakens the necessity half. Unlike the $n = 3$ case, the platform $\overline{x}$ may be a core point and yet not form a local equilibrium. For a concrete example, let $d = 1$ and define voter utility functions as follows:

$$\begin{aligned}
u_1(x) &= -3x - 6(x - 1/2)^4 \\
u_2(x) &= -x - 2(x - 1/2)^4 \\
u_3(x) &= -(x - 1/2)^4 \\
u_4(x) &= 2x - 4(x - 1/2)^4 \\
u_5(x) &= u_4(x).
\end{aligned}$$

It is straightforward to verify that voters 1 and 2 have ideal point at zero, voter 3 has ideal point at one half, and voters 4 and 5 have ideal point at one. Also, $\overline{x}$ is located at the ideal point of voter 3, which is the unique core point. At $\overline{x}$, the derivatives of the voters' utility functions are -3, -1, 0, 2, and 2, respectively, while $u_i''(1/2) = u_i'''(1/2) = 0$ for each voter i. In particular, $\sum_{i \in N}(D_{\hat{t}} u_i(\overline{x}))^3 = -12 < 0$, and condition (11) is fulfilled. It follows that arbitrarily small increases in A's platform will produce a probability of winning greater than one half, so $(\overline{x}, \overline{x})$ is not a local equilibrium — this despite the fact that increases in A's platform are worse for a majority of voters.

Another way Theorem 6 differs from Theorem 5 is that $\sum_{i \in N}(D_{\hat{t}} u_i(\overline{x}))^3 < 0$ is derived as a necessary condition for the existence of arbitrarily small profitable deviations along some line when $n = 3$. For larger n, the gain from deviating in (9) may be positive even when $\sum_{i \in N}(D_{\hat{t}} u_i(\overline{x}))^3 = 0$, for it may be that positive terms corresponding to coalitions of size four and higher dominate — a possibility not present when $n = 3$.

As a last remark, the theorem establishes conditions under which there exist arbitrarily small profitable deviations from $\overline{x}$ for an *arbitrary* odd number of voters. Thus, in the absence of negative definiteness of $D^2 U(\overline{x})$, equivalence of local equilibria under expected plurality and probability of winning will not hold generally even for very large n.

9 Global Equilibria

More to come.

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