Mixed Refinements of Shapley's Saddles and Weak Tournaments

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Abstract

We investigate refinements of two solutions, the saddle and the weak saddle, defined by Shapley (1964) for two-player zero-sum games. Applied to weak tournaments, the first refinement, the *mixed saddle,* is unique and gives us a new solution, generally lying between the GETCHA and GOTCHA sets of Schwartz (1972, 1986). In the absence of ties, all three solutions reduce to the usual top cycle set. The second refinement, the *weak mixed saddle,* is not generally unique, but, in the absence of ties, it is unique and coincides with the minimal covering set.

1 Introduction

Suppose each of two parties chooses a policy platform from a finite set of alternatives, and that the electorate decides between the chosen platforms by majority vote. Assume the parties care only about winning the election, not about the policies eventually implemented. What platforms will be adopted? The situation is inherently indeterminate: a pure strategy equilibrium exists only if there is a Condorcet winner, a platform unbeaten by any others in majority comparisons, and that cannot be taken for granted. Often, the set of Condorcet winners is empty. But there are other, arguably compelling, methods of formulating non-empty choice sets based on majority preferences, methods that choose the Condorcet winners when they exist yet yield non-empty choice sets when they do not.

Some central methods are the top cycle set of Schwartz (1972, 1986) and Miller (1977), the uncovered set of Fishburn (1977) and Miller (1980), and the minimal covering set of Dutta (1988). When the majority relation is a *tournament* (no two policy platforms are tied in pairwise comparisons), the top cycle set can be defined as follows: x is in the top cycle set if and only if, for every y, x beats some alternative that beats some alternative ... that beats some alternative that beats y. An alternative x is in the uncovered set if and only if, for every y, x beats y or beats some z that beats y. A *covering set* is a subset of policy platforms such that no alternative is "covered" within the set and every alternative outside the set is; and a minimal covering set is any covering set that includes no covering set as a proper subset. Dutta (1988) proves that there is exactly one minimal covering set, providing another method of formulating choice sets, generally much smaller than the top cycle set.

Miller (1977) provides one rationale for the top cycle set in terms of cooperative majority voting and one in terms of non-cooperative interaction, the latter showing that the top cycle set coincides with the possible outcomes of sophisticated voting under successive voting procedures. We provide a different type of positive foundation for the top cycle set related to a solution, due to Shapley (1964), for two-player zero-sum games. His solution is defined as follows in the context of party competition games. First, a *generalized saddle point* (GSP) is a subset, say K, of policy platforms for one party and a subset, say L, for the other such that (i) if platform x is not in K then there is some platform $y \in K$ that gives the first party a strictly higher payoff than x against every possible selection from L, and (ii) similarly for the second party. A *saddle* is defined as a minimal generalized saddle point — a GSP that includes no other GSP as a proper subset — and represents the platforms "playable" by the parties when these choice sets are common knowledge. Shapley proves that every finite two-player zero-sum game has a unique saddle.

After noting that the saddle of a party competition game is generally quite large: if ther are no majority ties, it necessarily encompasses every policy platform unless there is a Condorcet winner. We define a refinement, called the *mixed saddle,* motivated by the possibility that parties may use mixed strategies. Specifically, we replace (i) with (i$'$): if platform x is not in K then there is some probability distribution (a mixed strategy) with support in K that gives the first party a strictly higher expected payoff than x against every possible selection from L. Thus, we preserve the idea that K and L should represent the sets of playable pure strategies when these sets (but perhaps not the mixed strategies themselves) are common knowledge, while producing tighter predictions. Elsewhere, we establish the existence and uniqueness of the mixed saddle in a class of finite games including two-player zero-sum games, and, therefore, in party competition games.[1]

[1]See Duggan and Le Breton (1997) for a general treatment of solutions for strategic

We show below that, when there are no majority ties, the unique mixed saddle of the party competition game reduces to the top cycle set, providing the solution with a new positive foundation. The same type of positive rationale is given for the minimal covering set in Duggan and Le Breton (1996), where we use a refinement of the saddle, called the *weak saddle,* also proposed by Shapley (1964). It is defined as the saddle is, replacing (i) with (i$''$): if platform x is not in K then there is some platform $y \in K$ that gives the first party at least as high a payoff as x against every possible selection from L and a strictly higher payoff for at least one selection. Though the weak saddle is not unique in general party competition games, we show in our earlier paper that it is unique in the absence of ties and that, in this case, the unique weak saddle reduces to the minimal covering set.

We can refine the weak saddle the same way we did the saddle. Define the *weak mixed saddle* by replacing (i$''$) with (i$'''$): if platform x is not in K then there is some probability distribution (a mixed strategy) with support in K that gives the first party at least as high an expected payoff as x against every possible selection from L and a strictly higher expected payoff for at least one selection. We show that, in the absence of majority rule ties, mixing does not tighten the predictions of the weak saddle, contrary to the case of the saddle. That is, the weak mixed saddle is unique and reduces, as does the weak saddle, to the minimal covering set.

An open problem, posed by Moulin (1986), is the analysis of outcomes when ties are explicitly allowed, i.e., the majority relation is a *weak tournament.* Schwartz (1972, 1986) defines two versions of the top cycle set, called the GETCHA set and the GOCHA set, in this more general setting. The GETCHA set is the (unique) minimal set such that every platform in the set beats every platform outside in majority comparisons. A smaller choice set would be the minimal set such that no platform in the set is beaten by

games based on dominance concepts and their connections to rationalizability.

any platform outside, but this choice set is not well-defined, as there can be several such sets. The GOCHA set is the union of them. In general, the GOCHA set is a subset of the GETCHA set, and the two coincide with the usual top cycle set in the absence of ties. Our earlier uniqueness result for the mixed saddle holds for weak tournaments, and we prove here that the mixed saddle is generally nested between the GOCHA set and the GETCHA set. We show both inclusions may be strict. Thus, the mixed saddle gives us a new method, based on game-theoretic principles, of formulating choice sets from weak tournaments. However, we lose uniqueness of the weak saddle (this is shown in Duggan and Le Breton (1996)) and the mixed weak saddle when we go to weak tournaments, and we lose the equivalence of the two solutions.

In Section 2, we introduce notation for weak tournaments and, in this context, several well-known solutions. In Section 3, we define Shapley's (1964) saddle and weak saddle, and we present some basic results for weak tournaments. In Section 4, we define the mixed saddle and establish its relationships with several other solutions, including the GOCHA and GETCHA sets. Corollary 1 contains our positive foundation for the top cycle set of a tournament. We also define and analyze the weak mixed saddle. In Section 5, we present an exhaustive account of the relationships among our new solutions and the standard ones. An appendix contains numerous examples of weak tournaments to illustrate these relationships.

2 Preliminaries

Let X be a finite set of n alternatives, and let M be an asymmetric binary relation, or *weak tournament,* on X. It may be, for example, that xMy just when x beats y in a majority vote. Because we do not impose the assumption that M is total ($x \neq y$ implies either xMy or yMx), we allow for the possibility that x and y tie. M is a *tournament* if it is total as well as asymmetric.

There are many different ways to formulate choice sets based on a weak tournament. The most direct is the Condorcet method, which specifies the set $C(M) = \{x \in X | \neg\exists y \in X : yMx\}$ of M-maximal alternatives, i.e., the Condorcet winners, but $C(M)$ may be empty. An alternative would be to iteratively eliminate Condorcet losers, where x is a *Condorcet loser* in $Y \subseteq X$ if $x \in Y$ and, for all $y \in Y \setminus \{x\}$, yMx. Letting X^1 denote the set of alternatives remaining after eliminating the Condorcet loser in X (if any) and letting $k > 1$, define X^k as the elements of X^{k-1} remaining after deleting the Condorcet loser in X^{k-1} (if any). Because X is finite, this iterative process of elimination stops at some k^* with $X^{k^*} = X^{k^*+1}$. The result, which we denote X^*, is, of course, typically a large set.

Schwartz (1972, 1986) defines two choice sets that extend the Condorcet solution of a tournament and refine X^*. GETCHA(M) is the minimal set Y satisfying

$$\forall x \in Y, y \notin Y : xMy.$$

There is only one such set and it is non-empty, so the GETCHA set is well-defined. GOCHA(M) is union of minimal sets Z_k satisfying

$$\forall x \in Z_k, y \notin Z_k : \neg yMx.$$

The GOCHA set is always a non-empty subset of the GETCHA set, and equality of the solutions holds for tournaments. Deb (1977) and Schwartz (1986) prove the result, stated next, that each Z_k in the above definition contains an exhaustive M-cycle.

Proposition 1 *For all distinct $x, y \in Z_k$, there exist an integer l and $x_1, x_2, \ldots x_l \in Z_k$ such that $xMx_1Mx_2 \ldots Mx_l = y$.*

Another method, initiated by Fishburn (1977) and Miller(1980), is to choose the maximal elements of a "covering relation," appropriately defined.

We will follow McKelvey (1986), Dutta and Laslier (1997), and Peris and Subiza (1997) in extending Miller's definition to weak tournaments: given a subset $Y \subseteq X$ and $x, y \in Y$, *x covers y in Y* if (i) xMy, (ii) for all $z \in Y$, zMx implies zMy, and (iii) for all $z \in Y$, yMz implies xMz. The *uncovered set,* denoted UC(M), consists of the alternatives x for which there is no alternative y that covers x in X.

Dutta (1988) defines a refinement of the uncovered set in two steps. A *covering set* is a set Y of alternatives such that (i) no $x \in Y$ is covered in Y, and (ii) for all $y \notin Y$, there is some $x \in Y$ that covers y in $Y \cup \{y\}$. A minimal covering set is a covering set including no covering sets as a proper subset. Dutta (1988) proves that, when M is a tournament, there is exactly one minimal covering set, denoted MC(M), thus producing a well-defined choice set. Dutta and Laslier (1997) and Peris and Subiza (1997) prove that uniqueness of the minimal covering set carries over to weak tournaments.

Corresponding to a weak tournament M is a two-player zero-sum game, which we call the *M-game*. After indexing the alternatives $x_1, x_2, \ldots, x_n$, this game is defined as follows: player 1 chooses strategies x_i from X and player 2 chooses strategies x_j from X; player 1's payoff is 1 if $x_i M x_j$, -1 if $x_j M x_i$, and 0 otherwise; and player 2's payoff is negative one times player 1's. Let A be the payoff matrix of this game, where element a_i^j gives player 1's payoff if 1 chooses x_i and 2 chooses x_j. Of course, player 2's payoff is $-a_i^j$. We write a_i for the ith row of this matrix and a^j for the jth column. A mixed strategy for player 1 is a non-negative vector $p = (p_1, \ldots, p_n)$ with components summing to one, and a mixed strategy for player 2 is a non-negative vector $q = (q_1, \ldots, q_n)$ (we may think of this as a column vector), also with components summing to one. Player 1's payoff from the mixed strategy pair (p, q) is pAq, and player 2's payoff is $-pAq$.

Another approach to formulating choice sets, one generating interesting positive interpretations, is to apply game-theoretic solution concepts to the

M-game of a weak tournament. Given a set $Y \subseteq X$, let BR(Y) denote the set of pure strategy best responses to mixed strategies in Y. A set $K \times L$ of strategy profiles is *rationalizable* if K =BR(L) and L =BR(K). Bernheim (1984) and Pearce (1984) establish that there is a unique maximal rationalizable set, say $K \times K$. Letting $\mathrm{R}(M) = K$, this gives us a well-defined choice set. The following result illuminates the rationalizability solution and its close relationship with the iterative elimination of Condorcet losers.

Proposition 2

(i) $R(M) \subseteq X^*$.

(ii) If M *is a tournament,* $R(M) = X^*$.

Proof: (i) Clearly, $R(M) \subseteq X^1$. If a point in $X^1 \setminus X^2$ is a best response to some mixed strategy, that mixed strategy must put probability one on $X \setminus X^1$, and no alternative in this set is rationalizable. Therefore, $R(M) \subseteq X^2$. An induction argument based on this principle establishes $R(M) \subseteq X^*$. (ii) Take any $x \in X^*$. Since x is not a Condorcet loser in X^*, there is some $x^1 \in X^*$ such that $\neg x^1 M x$. Since M is assumed to be a tournament, $x M x^1$ and x is a best response to x^1. Similarly, there is some $x^2 \in X^*$ such that x^1 is a best response to x^2, and so on. Thus, x is rationalizable. ∎

Laffond, Laslier, and Le Breton (1993) prove that, when M is a tournament, the M-game has a unique mixed strategy equilibrium. They define the *bipartisan set* as K, where $K \times K$ is the support of that equilibrium. In weak tournaments, of course, there may be multiple mixed strategy equilibria. Dutta and Laslier (1997) note that there is a unique mixed strategy equilibrium with maximal support, say $K \times K$, and they define the *essential set,* denoted ES(M), by ES(M) $= K$. The next proposition records some well-known inclusion relationships among these solutions.

Proposition 3

(i) $ES(M) \subseteq MC(M) \subseteq UC(M) \subseteq GETCHA(M) \subseteq X^*$.

(ii) $ES(M) \cap GOCHA(M) \neq \emptyset$.

(iii) $GOCHA(M) \subseteq GETCHA(M) \cap R(M)$.

(iv) If M is a tournament, $GOCHA(M) = GETCHA(M)$.

Proof: (i) The first inclusion is proved by Dutta and Laslier (1997), and the second holds by definition. The third follows because the alternatives in $X \backslash \text{GETCHA}(M)$ are covered by the elements of GETCHA(M). The fourth holds because xMy for all $x \in X^*$ and all $y \notin X^*$.

(ii) Let Z_k be any component of the GOCHA set, and let (p, q) be any mixed strategy equilibrium of the M_{Z_k}-game, where M_{Z_k} is the restriction of M to $Z_k \times Z_k$. Thus, $pAq = 0$. If (p, q) is not an equilibrium of the M-game, some player, say player 1, has a mixed strategy $p' \in \Delta(X)$ such that $p'Aq > 0$. If $x \in Z_k$ and $y \notin Z_k$, then $\neg yMx$ by the definition of Z_k, so $p'(Z_k) > 0$. Defining $p'' \in \Delta(Z_k)$ by $p''(x) = p'(x)/p'(Z_k)$, we have $p''Aq > 0$, a contradiction.

(iii) The first inclusion is proved by Schwartz (1986), and the relationship between the GOCHA set and R(M) follows from Proposition 1.

(iv) This is proved by Schwartz (1986). ∎

In Section 5, we investigate the inclusion relationships among these solutions and several more, defined in the sequel.

3 Saddles and Weak Saddles

Shapley (1964) defined a new solution, the "saddle," for arbitrary two-player zero-sum games. We define it next for the special case of M-games, using the following notation. Given vectors $u, v \in \Re^n$ and a subset $K \subseteq X$, $u \gg_K v$ if $u_i > v_i$ for all i with $x_i \in K$.

Definition 1 *A Generalized Saddle Point (GSP) is a product $K \times L$ of non-empty subsets of X such that*

$$\forall x_i \notin K, \exists x_k \in K : a_k \gg_L a_i$$

and

$$\forall x_j \notin L, \exists x_l \in L : a^j \gg_K a^l.$$

A GSP that includes no other GSP as a proper subset is a Saddle.

It is clear that every M-game has at least one saddle: the set $X \times X$ is itself a GSP; if it is not a saddle then it includes a GSP, $K \times L$, as a proper subset; if this is not a saddle, it includes a GSP, $K' \times L'$, as a proper subset; and so on. By finitude, this process must stop at a saddle. Shapley proves more: every finite two-player zero-sum game has a *unique* saddle. Let $S(M) = K$, where $K \times K$ is the unique saddle of the M-game. As the next result shows, however, the saddle often embraces the entire set of alternatives.

Theorem 1 *If $S(M) \neq X$ then $C(M) \neq \emptyset$. If M is a tournament and $S(M) \neq X$ then $S(M) = C(M)$.*

Proof: Take any $x_i \notin S(M)$. Since $x_i \notin S(M)$, there exists $x_k \in S(M)$ such that $a_k \gg_{S(M)} a_i$. Thus, for all $x_j \in S(M)$, (i) $a_k^j \geq 0$ and (ii) $a_i^j \leq 0$. The former condition states that, for all $x_j \in S(M)$, $\neg x_j M x_k$. The latter condition tells us that, for all $x_j \in S(M)$, $\neg x_i M x_j$. Since x_i was an arbitrary alternative outside $S(M)$ and $x_k \in S(M)$, we have $x_k \in C(M)$.

If M is a tournament then conditions (i) and (ii) imply that, for all $x_j \neq x_k$, $x_k M x_j$, implying $S(M) = \{x_k\} = C(M)$. ∎

The next example shows that $C(M) \neq \emptyset$ in the first part of Theorem 1 cannot be strengthened to $S(M) = C(M)$ for general weak tournaments. It is easily checked that $S(M) = \{x_1, x_2, x_3\}$, despite $C(M) = \{x_1, x_2\}$.

	x_1	x_2	x_3	x_4
x_1	0	0	1	1
x_2	0	0	0	1
x_3	-1	0	0	1
x_4	-1	-1	-1	0

The next result establishes that, accordingly, the uncovered is generally a subset of the saddle.

Theorem 2 *$UC(M) \subseteq S(M)$.*

Proof: Take any $x_h \notin S(M)$ and $x_i \in S(M)$ such that $x_i \gg_{S(M)} x_h$, implying $x_i M x_h$ and, for all $x_j \in S(M)$, $a_h^j \leq 0$. As in the proof of Theorem 1, $x_i \in C(M)$. Now take any $x_j \in X$ such that $x_h M x_j$. Since $a_h^j = 1$, we have $x_j \notin S(M)$, so there exists $x_k \in S(M)$ such that $x_k \gg_{S(M)} x_j$, implying $a_j^i \leq 0$. If not $x_i M x_j$, we have $a_j^i = 0$. But then $a_k^i > a_j^i = 0$ implies $a_k^i = 1$, or equivalently, $x_k M x_i$, contradicting $x_i \in C(M)$. Therefore, $x_i M x_j$, and we conclude that x_i covers x_h. ∎

Before moving on from the saddle, we offer two examples of tournaments illustrating the relationships between S(M), R(M), and X^*: even for tournaments, S(M) may be a proper subset or superset of the other solutions.

	x_1	x_2	x_3	x_4
x_1	0	1	-1	1
x_2	-1	0	1	1
x_3	1	-1	0	1
x_4	-1	-1	-1	0

Here, $\mathrm{S}(M) = \{x_1, x_2, x_3, x_4\}$, while $\mathrm{R}(M) = X^* = \{x_1, x_2, x_3\}$. Below, the story is different.

	x_1	x_2	x_3	x_4
x_1	0	1	1	1
x_2	-1	0	-1	1
x_3	-1	1	0	-1
x_4	-1	-1	1	0

Here, $\mathrm{S}(M) = \{x_1\}$ while $\mathrm{R}(M) = X^* = \{x_1, x_2, x_3, x_4\}$. Note that $\mathrm{C}(M) \neq \emptyset$ in the latter example, as is necessary by Theorem 1 if $\mathrm{S}(M)$ is to be a proper subset of X.

Shapley also defines a refinement of the saddle using a weaker notion of dominance. Given vectors $u, v \in \Re^n$ and a subset $K \subseteq X$, $u >_K v$ if $u_i \geq v_i$

for all i with $x_i \in K$, at least one inequality strict.

Definition 2 *A Weak Generalized Saddle Point (WGSP) is a product $K \times L$ of non-empty subsets of X such that*

$$\forall x_i \notin K, \exists x_k \in K : a_k >_L a_i$$

and

$$\forall x_j \notin L, \exists x_l \in L : a^j >_K a^l.$$

A WGSP that includes no other WGSP as a proper subset is a Weak Saddle.

The same argument for the existence of a saddle delivers the existence of a weak saddle.[2] Shapley notes that the weak saddle is not generally unique, but Duggan and Le Breton (1996) establish uniqueness for a subclass of two-player zero-sum games that includes M-games when M is a tournament. In that paper, we show that, for tournaments, the weak saddle refines the saddle significantly: it reduces to Dutta's (1988) minimal covering set. It is clear that the saddle generally includes a weak saddle, because the former is itself a WGSP. In Section 5, however, we will see that it need not include all weak saddles.

The next result establishes the relationship between weak saddles and the essential set. With Proposition 3 and Theorem 2, it also implies that every weak saddle has non-empty intersection with the saddle.

Theorem 3 *If $K \times L$ is a weak saddle then $K \cap ES(M) \neq \emptyset$ and $L \cap ES(M) \neq \emptyset$.*

Proof: Consider the M-game restricted to $K \times L$, i.e., where player 1's strategy set is K and player 2's is L, with payoffs derived from M. There

[2] Duggan and Le Breton (1997) extend Shapley's definitions to infinite games and establish existence of saddles, weak saddles, and a family of other solutions with a similar argument, but using Zorn's lemma instead of finitude.

exists at least one mixed strategy equilibrium (p,q) of the restricted game. If it is not an equilibrium of the M-game, then some player, say player 1, has a strategy $x_i \notin K$ such that $a_i \cdot q > pAq$. But there exists $x_k \in K$ such that $a_k >_L a_i$, implying $a_k \cdot q \geq a_i \cdot q > pAq$, a contradiction. ∎

The weak saddles also have non-empty intersection with the GOCHA set.

Theorem 4 *If $K \times L$ is a weak saddle then $K \cap GOCHA(M) \neq \emptyset$ and $L \cap GOCHA(M) \neq \emptyset$.*

Proof: Suppose that $K \times L$ is a weak saddle with GOCHA$(M) \cap L = \emptyset$. Take $x_j \in$GOCHA(M) as follows: if $K\cap$GOCHA$(M) \neq \emptyset$, let $x_j \in K\cap$GOCHA(M); otherwise, x_j may be an arbitrary element of GOCHA(M). In either case, $x_j \notin L$. Using the definition of the weak saddle, take any $x_l \in L$ such that $a^j >_K a^l$.

We claim that, for all $x_h \in K$, $a_h^l \leq 0$. If $x_h \in K\backslash$GOCHA(M), $\neg x_h M x_j$ follows from the definition of the GOCHA set, giving us $a_h^j \leq 0$, and then $a^j >_K a^l$ implies $a_h^l \leq 0$. Now suppose $x_h \in$GOCHA$(M) \cap K$ yet $a_h^j = 1$. Note that $x_h \notin L$ by assumption, and $x_j \in K$ by construction. Since $x_h \notin L$, there exists some $x_m \in L$ such that $a^h >_K a^m$. And since $x_j \in K$ and $a_j^h = -1$, it must be that $-1 = a_j^h \geq a_j^m$, implying $a_j^m = -1$, or $x_m M x_j$. But this implies $x_m \in$GOCHA(M), contradicting $x_m \in L$ and $L\cap$GOCHA$(M) = \emptyset$. Therefore, $a_h^j \leq 0$, and then $a^j >_K a^l$ implies $a_h^l \leq 0$.

We now claim that, for all $x_h \notin K$, $a_h^l \leq 0$. Take any $x_k \in K$ such that $a_k >_L a_h$. We have shown that $a_k^l \leq 0$, which implies $a_h^l \leq 0$, as claimed.

This establishes $a_h^l \leq 0$ for all x_h, implying $x_l \in C(M)$, but then $x_l \in$GOCHA(M), contradicting GOCHA$(M) \cap L = \emptyset$. ∎

In contrast, the GETCHA set is large enough to bound all weak saddles.

Theorem 5 *If $K \times L$ is a weak saddle then $K \cup L \subseteq GETCHA(M)$.*

Proof: From Proposition 3 and Theorem 3, it follows that $K' = K \cap$ GETCHA$(M) \neq \emptyset$ and $L' = L \cap$GETCHA$(M) \neq \emptyset$. It suffices to show that $K' \times L'$ is a WGSP. Take any $x_i \notin K'$. If $x_i \notin$GETCHA(M), note that $a_i^h = -1$ for all $x_h \in$GETCHA(M), and therefore for all $x_h \in L'$. Taking any $x_l \in L'$, we also have $a_l^l = 0 > -1 = a_i^l$, which implies $a_l >_{L'} a_i$. If $x_l \in K$, and therefore $x_l \in K'$, the case is finished. Otherwise, there is some $x_k \in K$ such that $a_k >_L a_l$, implying $a_k >_{L'} a_i$.

If $x_i \in$GETCHA(M) then $x \notin K$, so there is some $x_{k_1} \in K$ such that $a_{k_1} >_L a_i$. Thus, there is some $x_h \in L$ such that $a_{k_1}^h > a_i^h$. That $x_h \in L'$ follows because otherwise $x_h \notin$GETCHA(M), implying $x_i M x_h$, i.e., $a_i^h = 1$, a contradiction. Therefore, $a_{k_1} >_{L'} a_i$. If $x_{k_1} \in K'$, the case is finished. Otherwise, the argument can be repeated to find $x_{k_2} \in K$ such that $a_{k_2} >_{L'} a_{k_1} >_{L'} a_i$. If $x_{k_2} \in K'$, again the case is finished. Otherwise, the argument can be repeated. Because X is finite, this process must stop at some $x_k \in K'$ such that $a_k >_{L'} a_i$. ∎

4 Mixed Refinements of the Saddle and Weak Saddle

We now consider a refinement, introduced by Duggan and Le Breton (1997), of Shapley's saddle, based on the possibility that players may use mixed strategies. In light of this possibility, player 1 may reject strategy x_i because it is strictly dominated by a *mixed* strategy — not necessarily a pure one, as would be called for in Shapley's definition of GSP. This creates more generalized saddle points and a tighter notion of equilibrium. The solution is defined next for M-games. Denote by $\Delta(K)$ the set of mixed strategies with support in K.

Definition 3 *A Mixed Generalized Saddle Point (MGSP) is a product* $K \times L$

of non-empty subsets of X such that

$$\forall x_i \notin K, \exists p \in \Delta(K) : pA \gg_L a_i$$

and

$$\forall x_j \notin L, \exists q \in \Delta(L) : a^j \gg_K Aq.$$

A MGSP that includes no other MGSP as a proper subset is a Mixed Saddle.

Existence of a mixed saddle follows from the same sort of argument as that used for the saddle and the weak saddle. It is clear that, because the saddle is a MGSP, it includes a mixed saddle. Duggan and Le Breton (1997) prove uniqueness of the mixed saddle for a class of games including finite two-player zero-sum games, which include the M-games. Thus, the mixed saddle is included in the saddle. Let $\mathrm{MS}(M) = K$, where $K \times K$ is the mixed saddle of the M-game. The next result establishes the connection between the mixed saddle and rationalizability, and it reveals an interesting aspect of rationalizable sets: the mixed saddle corresponds to the *unique minimal* rationalizable set.

Theorem 6 *The mixed saddle, $MS(M) \times MS(M)$ is a rationalizable set, and if $K \times L$ is a rationalizable set then $MS(M) \subseteq K \cap L$.*

Proof: By Myerson's (1991) Theorem 1.6, if $K \times L$ is a rationalizable set then it is a MGSP, and therefore it includes the mixed saddle. As well, if the mixed saddle is not rationalizable then there is some $x \in \mathrm{MS}(M)$ that is a best response to no mixed strategy with support in $\mathrm{MS}(M)$. But, again using Myerson's theorem, then $(\mathrm{MS}(M) \setminus \{x\}) \times (\mathrm{MS}(M) \setminus \{x\})$ is a MGSP, contradicting minimality of the mixed saddle. ∎

We can now compare the mixed saddle to the essential set. Note that the next result, with Theorem 3, implies that the mixed saddle has non-empty intersection with every weak saddle.

Theorem 7 *ES*$(M) \subseteq$*MS*(M).

Proof: Let MS$(M) = K$, and let (p, p) denote a symmetric mixed strategy equilibrium of the M_K-game, where M_K is the restriction of M to $K \times K$. Note as well that (p, p) is a mixed strategy equilibrium of the M-game. Now take any other equilibrium of the M-game, which, using interchangeability, we can take to be symmetric, say (q, q). Then, again by interchangeability, (q, p) is an equilibrium, and it follows that each strategy x_j with $q_j > 0$ is a best response to p. Since $K \times K$ is a best response set, $x_j \in K$. Thus, the support of every mixed strategy used in an equilibrium is included in K, implying ES$(M) \subseteq$MS(M). ∎

That the essential set may be a proper subset of the mixed saddle is demonstrated in the appendix. Though the relation M defined there is not a tournament, it is easy to find examples of tournaments in which the essential set is a proper subset of the mixed saddle. The following theorem indicates why this is so. It establishes the relationships between the mixed saddle and Schwartz's top cycle sets. We will show in Section 5 that Theorem 7 does *not* follow from the first inclusion of Theorem 8, i.e., we show that there are weak tournaments for which the essential set is not a subset of the GOCHA set.

Theorem 8 *GOCHA*$(M) \subseteq$*MS*$(M) \subseteq$*GETCHA*(M).

Proof: First, GOCHA$(M) \subseteq$MS$(M) = K$. Let GOCHA$(M) = \bigcup_k Z_k$, where each Z_k is as in the definition of the GOCHA set, and suppose there exists $x_i \in Z_k \backslash K$ for some k. Thus, there exists $p \in \Delta(K)$ such that $pA \gg_K a_i$, and it follows that $a_i^j = -1$ or 0 for all $x_j \in K$. If $a_i^j = -1$ for some $x_j \in K$ then, by construction of Z_k, $x_j \in Z_k$. Using Proposition 1, we could then take a sequence $x_{i_1}, x_{i_2}, \ldots, x_{i_l} \in Z_k$ such that $x_i M x_{i_1} M x_{i_2} M \ldots M x_{i_l} = x_j$. Thus, $a_i^{i_1} = 1$, which implies $x_{i_1} \notin K$. The same argument then establishes that $x_{i_2} \notin K$, and so on, with the eventual conclusion that $x_j \notin K$, a contradiction. Thus, $a_i^j = 0$ for all $x_j \in K$, and then $pA \gg_K a_i$ implies that $pA \gg_K 0$. Now consider the weak tournament M_K, the restriction of M to $K \times K$, and the M_K-game. Applying the Minmax Theorem (see Gale (1960), Theorem 6.6) to this symmetric zero-sum game, player 2 has an optimal strategy $q \in \Delta(K)$ that limits player 1 to a non-positive expected payoff. But $pA \gg_K 0$ implies $pAq > 0$, a contradiction.

Second, MS(M) $\subseteq$GETCHA(M) $= K$. It suffices to show that K is a MGSP. Take any $x_i \notin K$ and the mixed strategy, $\overline{p}$, that selects strategies from K with equal probability. For all $x_j \in K$, $x_j M x_i$, or equivalently, $a_i^j = -1$. Thus,

$$\overline{p} \cdot a^j = \frac{1}{|K|} \sum_{x_k \in K} a_k^j > -1 = a_i^j,$$

where the inequality follows from $a_k^j \geq -1$ and $a_j^j = 0$. This gives us $\overline{p}A \gg_K a_i$. ∎

In the appendix, we show that the inclusions of Theorem 8 may be strict. As in Proposition 3, GETCHA(M) =GOCHA(M) when M is a tournament, giving us the following corollary.

Corollary 1 *If M is a tournament then GETCHA(M) =GOCHA(M) =MS(M).*

As in Proposition 3, the uncovered set is always a subset of the GETCHA set, and it (and therefore the minimal covering set) is known to sometimes be a proper subset when M is a tournament. Using the above corollary, this observation carries over to the mixed saddles of tournaments. We show in the appendix that, for general weak tournaments, the mixed saddle may be a proper subset of the minimal covering set (and therefore the uncovered set). Thus, there is no general inclusion relationship between the mixed saddle and either of these two solutions. Note, however, that Proposition 3 and

Theorem 7 yield $MC(M) \cap MS(M) \neq \emptyset$ (and therefore $UC(M) \cap MS(M) \neq \emptyset$) for all weak tournaments.

We can refine Shapley's weak saddle as we did the saddle, similarly allowing players to use mixed strategies.

Definition 4 *A Weak Mixed Generalized Saddle Point (WMGSP) is a product $K \times L$ of non-empty subsets of X such that*

$$\forall x_i \notin K, \exists p \in \Delta(K) : pA >_L a_i$$

and

$$\forall y_j \notin L, \exists q \in \Delta(L) : a^j >_K Aq.$$

A WMGSP that includes no other WMGSP as a proper subset is a Weak Mixed Saddle.

Existence of a weak mixed saddle follows from the same argument given for the saddle. The weak mixed saddle inherits the non-uniqueness of the weak saddle: Duggan and Le Breton (1996) provide an example of an M-game with multiple weak saddles that also happen to be weak mixed saddles. It is clear that, because every weak saddle is a WMGSP, every weak saddle includes a weak mixed saddle. In Section 5, we show that this inclusion can be strict, even when the solutions are unique, so the weak mixed saddle refines the weak saddle, in a sense. However, we also give an example of an M-game in which the weak saddle is unique and there is a weak mixed saddle disjoint from it. Thus, the weak mixed saddle offers only a partial refinement of the weak saddle.

The next result establishes that every weak mixed saddle has non-empty intersection with the essential set. The proof parallels that of Theorem 3 and is omitted.

Theorem 9 *If $K \times L$ is a weak mixed saddle then $K \cap ES(M) \neq \emptyset$ and $L \cap ES(M) \neq \emptyset$.*

We also note that the GETCHA set includes all of the weak mixed saddles. The proof of the result parallels that of Theorem 5 and is omitted. In contrast, we show in the appendix that it is possible for a game to have a unique weak mixed saddle that is disjoint from the GOCHA set.

Theorem 10 *If $K \times L$ is a weak mixed saddle then $K \cup L \subseteq GETCHA(M)$.*

Like the weak saddle, the weak mixed saddle is unique when M is a tournament. In fact, as the next result shows, the two concepts are equivalent for tournaments.

Theorem 11 *If M is a tournament then the weak mixed saddle of the M-game is unique and equal to the weak saddle.*

Proof: Every WGSP is a WMGSP. Now take a WMGSP $K \times L$ and $x_i \notin K$. Let $p \in \Delta(K)$ satisfy $pA >_L a_i$. If $x_i \notin L$, then let $x_l \in L$ be any alternative satisfying $p \cdot a^l > a_i^l$. Thus, there exists some $x_k \in K$ with $p_k > 0$ such that $a_k^l > a_i^l$. We claim that $a_k >_L a_i$. Take any $x_j \in L$. If $a_i^j = -1$ then $a_k^j \geq a_i^j$. If $a_i^j = 1$ then, since $p_k > 0$ and $p \cdot a^l >_L a_i$, it follows that $a_k^j = 1 \geq a_i^j$. If $a_i^j = 0$ then, because M is a tournament, $x_i = x_j \in L$, contrary to our assumptions. Hence, $a_k >_L a_i$.

If $x_i \in L$ then, because $p \cdot a^i > a_i^i = 0$, there is some $x_k \in K$ such that $p_k > 0$ and $a_k^i = 1$. Taking any $x_j \in L$, as above, $a_k^j \geq a_i^j$ whenever $a_i^j = -1$ or $a_i^j = 1$. If $a_i^j = 0$ then, since M is a tournament, $x_j = x_i$, and, by our choice of x_k, $a_k^j = 1 > 0 = a_i^j$. Hence, $a_k >_L a_i$. Therefore, every MGSP is a WGSP. ∎

Thus, unlike the mixed saddle, the weak mixed saddle refinement does not tighten the predictions of the weak saddle in tournaments. Because Duggan and Le Breton (1996) prove the equivalence of the weak saddle with the minimal covering set for tournaments, we also obtain equivalence for the weak mixed saddle.

5 Logical Connections

In this section, we trace out the logical relationships among the solutions defined above. Table 1 summarizes the results for weak tournaments, where we use $\subset$ to indicate that the row solution is generally a subset (sometimes proper) of the column solution; we use $\emptyset$ to indicate that the two solutions can be disjoint; and we use $\cap$ to indicate that the row and column solutions generally have non-empty intersection but are non-nested. (Often, we can give an example in which one solution is a proper subset of the other and an example exhibiting the opposite strict inclusion.) Because the weak saddle and weak mixed saddle are not generally unique, the results involving them can be somewhat complicated — they are explained in more detail below.

	WMS	WS	MC	UC	GOCHA	MS	GETCHA	R	S	X^*
ES	$\cap^{1}$	$\cap^{2}$	$\subset^{3}$	$\subset^{4}$	$\cap^{5}$	$\subset^{6}$	$\subset^{7}$	$\subset^{8}$	$\subset^{9}$	$\subset^{10}$
WMS		$\cap^{11}$	$\cap^{12}$	$\cap^{13}$	$\emptyset^{14}$	$\cap^{15}$	$\subset^{16}$	$\cap^{17}$	$\cap^{18}$	$\subset^{19}$
WS			$\cap^{20}$	$\cap^{21}$	$\cap^{22}$	$\cap^{23}$	$\subset^{24}$	$\cap^{25}$	$\cap^{26}$	$\subset^{27}$
MC				$\subset^{28}$	$\cap^{29}$	$\cap^{30}$	$\subset^{31}$	$\cap^{32}$	$\subset^{33}$	$\subset^{34}$
UC					$\cap^{35}$	$\cap^{36}$	$\subset^{37}$	$\cap^{38}$	$\subset^{39}$	$\subset^{40}$
GOCHA						$\subset^{41}$	$\subset^{42}$	$\subset^{43}$	$\subset^{44}$	$\subset^{45}$
MS							$\subset^{46}$	$\subset^{47}$	$\subset^{48}$	$\subset^{49}$
GETCHA								$\cap^{50}$	$\cap^{51}$	$\subset^{52}$
R									$\cap^{53}$	$\subset^{54}$
S										$\cap^{55}$

Table 1

In the rest of this section, we verify the entries of Table 1, referring to cells by their numbers. Examples can be found in the appendix.

1. Theorem 9. In tournaments, the weak mixed saddle coincides with the minimal covering set, which is known to sometimes be a strict superset of the essential set. See Example 1 for an M-game in which the essential set is not a subset of the unique weak mixed saddle.

2. Theorem 3. For non-nestedness, same as 1.

3. Proposition 3. Strict inclusions are well-known in the tournament literature.

4. From 3 and 28.

5. Proposition 3. It is well-known in the tournament literature that ES(M) may be a proper subset of GOCHA(M). See Example 1 for the opposite strict inclusion.

6. Theorem 7. See Example 1 for strict inclusion.

7. From 3 and 31.

8. From 6 and 47.

9. From 6 and 48.

10. From 6 and 49.

11. Every weak saddle is a WMGSP, and so includes a weak mixed saddle. See Example 2 for an M-game with a unique weak saddle and a unique weak mixed saddle, the latter a proper subset of the former. See Example 3 for an M-game with a unique weak saddle and a weak mixed saddle disjoint from it.

12. Since MC(M) is a WMGSP, there is always a weak mixed saddle included within the minimal covering set. See Example 1 for an M-game with a unique weak mixed saddle that is a proper subset of MC(M). However, see Example 4 for an M-game with a weak mixed saddle that is not included in MC(M).

13. From 12 and 28. For non-nestedness, same as 12.

14. See Example 5 for an M-game with a unique weak mixed saddle that is disjoint from the GOCHA set.

15. Theorem 7 and 9. See Example 1 for an M-game in which the mixed saddle is a proper superset of the unique weak mixed saddle, and see Example 6 for a weak mixed saddle that is not included within the mixed saddle.

16. Theorem 10. Examples of strict set inclusion are well known in the

literature on tournaments, where the weak mixed saddle is equivalent to the minimal covering set.

17. From 1 and 8. Using Proposition 2, examples of tournaments in which $R(M)$ is a proper superset of the weak mixed saddle, equivalent to the minimal covering set, are easily found. See Example 6 for a weak tournament with a weak mixed saddle that is not a subset of $R(M)$.

18. From 15 and 48. See Example 6 for a weak tournament with a weak mixed saddle that is not a subset of the saddle.

19. From 16 and 52.

20. Same as 12.

21. From 20 and 28.

22. Theorem 4. In tournaments, the weak saddle coincides with the minimal covering set, which is known to sometimes be a strict subset of GOCHA(M). See Example 4 for an M-game with a weak saddle that is not included in GOCHA(M).

23. Theorems 7 and 3. In tournaments, the weak saddle coincides with the minimal covering set and the mixed saddle coincides with the GOCHA and GETCHA sets, and the former is known to sometimes be a proper subset of the latter. See Example 7 for an M-game in which the unique weak saddle is a strict superset of the mixed saddle.

24. Theorem 5. Examples of strict inclusion are well known in the tournaments literature, where the weak saddle is equivalent to the minimal covering set.

25. From 2 and 8. In tournaments, $R(M) = X^*$ and the weak saddle is the minimal covering set, and the former can clearly be a proper superset of the latter. See Example 7 for the opposite strict inclusion.

26. Since $S(M)$ is a WGSP, there is always a weak saddle included within the saddle. Moreover, from 23 and 48, every weak saddle has non-empty intersection with the saddle. See Example 6 for an M-game with a weak

saddle that is not included in the saddle.

27. From 24 and 52.

28. Proposition 3. Strict inclusion is well-known in the literature on tournaments.

29. From 3 and 5. Examples of tournaments in which MC(M) is a proper subset of the GOCHA set are well-known. See Example 8 for the opposite strict inclusion.

30. From 3 and 6. Using Theorem 8, examples of tournaments in which MC(M) is a proper subset of MS(M) are easily found. See Example 8 for the opposite strict inclusion.

31. Proposition 3. Strict inclusion is well-known in the literature on tournaments.

32. From 3 and 8. Examples of tournaments in which MC(M) is a proper subset of $R(M)$ are easily found. See Example 8 for the opposite strict inclusion.

33. Theorem 2. Using Theorem 1, examples of tournaments exhibiting strict inclusion are easily found.

34. From 28 and 40.

35. From 4 and 5. Examples of tournaments in which UC(M) is a proper subset of GOCHA(M) are easily found. See Example 1 for the opposite strict inclusion.

36. From 4 and 6. Using Theorem 8, examples of tournaments in which MS(M) is a proper subset of UC(M) are easily found. See Example 8 for the opposite strict inclusion.

37. Proposition 3. Strict inclusion is well-known in the literature on tournaments.

38. From 4 and 8. Using Proposition 1, examples of tournaments in which UC(M) is a proper subset of $R(M)$ are easily found. See Example 7 for the opposite strict inclusion.

39. Theorem 2. Using Theorem 1, examples of tournaments in which the uncovered set is proper subset of the saddle are easy to find.

40. From 37 and 52.

41. Theorem 8. See Example 1 for strict inclusion.

42. Proposition 3. See Example 1 for strict inclusion.

43. From 41 and 47.

44. From 41 and 48.

45. From 42 and 52.

46. Theorem 8. See Example 8 for strict inclusion.

47. Theorem 6. Examples of tournaments exhibiting strict inclusion are easily found using Proposition 2, or see Example 8 for a weak tournament.

48. The saddle is a MGSP. See Example 8 for strict inclusion.

49. From 47 and 54.

50. From 7 and 8. Using Proposition 2, examples of tournaments in which the GETCHA set is a proper subset of $R(M)$ are easily found. See Example 7 for the opposite strict inclusion.

51. From 7 and 9. Using Theorem 1, examples of tournaments exhibiting the strict inclusion are easily found. See Example 9 for an M-game in which the GETCHA set is a proper superset of the saddle.

52. Proposition 3. Examples of tournaments exhibiting strict inclusion are easily found.

53. From 8 and 9. Section 3 contains examples of both strict inclusions.

54. Proposition 2. See Example 6 for proper inclusion.

55. From 9 and 10. Section 3 contains examples of both strict inclusions.

For the special case of tournaments, the relationships among these solutions are drastically simplified. They are recorded in the Table 2.

	WMS	WS	MC	UC	GOCHA	MS	GETCHA	R	S	X^*
ES	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
WMS		$=$	$=$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
WS			$=$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
MC				$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
UC					$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$
GOCHA						$=$	$=$	$\subset$	$\subset$	$\subset$
MS							$=$	$\subset$	$\subset$	$\subset$
GETCHA								$\subset$	$\subset$	$\subset$
R									$\cap$	$=$
S										$\cap$

Table 2

The entries of Table 2 that are not well-known follow from Proposition 2, Corollary 1, Theorem 11, and Duggan and Le Breton's (1996) theorem on the equivalence of the weak saddle and the minimal covering set of a tournament.

A Appendix

A.1 Example 1

	x_1	x_2	x_3
x_1	0	1	0
x_2	-1	0	1
x_3	0	-1	0

Here, MC(M) =UC(M) =GETCHA(M) $=$ $R(M)$ =MS(M) $= \{x_1, x_2, x_3\}$; ES(M) $= \{x_1, x_3\}$; GOCHA(M) $= \{x_1\}$; and the unique weak saddle and weak mixed saddle is $\{x_1, x_2\} \times \{x_1, x_2\}$.

A.2 Example 2

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	1	1	-1	-1	0
x_2	-1	0	1	1	-1	1
x_3	-1	-1	0	1	1	0
x_4	1	-1	-1	0	1	1
x_5	1	1	-1	-1	0	0
x_6	0	-1	0	-1	0	0

Here, the unique weak saddle (indeed, the minimal covering set) is $X \times X$, and the unique weak mixed saddle is $\{x_1, x_2, x_3, x_4, x_5\} \times \{x_1, x_2, x_3, x_4 x_5\}$. Let $K \times L$ be any weak saddle, and note that $|K| > 1$. There are fifteen cases, the analysis of each relying on the external stability property of the weak saddle.

Case 1. $x_1, x_2 \in K$. This implies $x_5 \in L$, giving us two subcases.

1.1 $x_3 \in K \Rightarrow x_1 \in L \Rightarrow x_4 \in K \Rightarrow x_2, x_4 \in L \Rightarrow x_5 \in K \Rightarrow x_3 \in L \Rightarrow x_6 \in K \Rightarrow x_6 \in L$. Thus, $K = L = X$.

1.2 $x_4 \in K \Rightarrow x_2, x_4 \in L \Rightarrow x_3, x_5 \in K \Rightarrow x_3 \in L \Rightarrow x_6 \in K \Rightarrow x_6 \in L$. Thus, $K = L = X$.

Case 2. $x_1, x_3 \in K$. This implies $x_1 \in L$, giving us two subcases.

2.1 $x_4 \in K \Rightarrow x_2, x_4 \in L \Rightarrow x_2 \in K$. By 1, $K = L = X$.

2.2 $x_5 \in K \Rightarrow x_3, x_4 \in L \Rightarrow x_2 \in K$. By 1, $K = L = X$.

Case 3. $x_1, x_4 \in K$. This implies $x_4 \in L$, giving us two subcases.

3.1 $x_2 \in K$. By 1, $K = L = X$.

3.2 $x_3 \in K$. By 2, $K = L = X$.

Case 4. $x_1, x_5 \in K$. This implies $x_4 \in L$, giving us two subcases, just as in Case 3.

Case 5. $x_1, x_6 \in K$. This implies $x_4 \in L$, giving us two subcases, just as in Case 3.

Case 6. $x_2, x_3 \in K$. This implies $x_1 \in L$, giving us two subcases.

6.1 $x_4 \in K \Rightarrow x_2 \in L \Rightarrow x_5 \in K \Rightarrow x_3 \in L \Rightarrow x_1 \in K$. By 1, $K = L = X$.

6.2 $x_5 \in K \Rightarrow x_3 \in L \Rightarrow x_1 \in K$. By 1, $K = L = X$.

Case 7. $x_2, x_4 \in K$. This implies $x_2 \in L$, implying $x_1 \in K$ or $x_5 \in K$. In the first subcase, 1 applies. In the second, we have $x_3 \in L$, implying $x_1 \in K$, and again 1 applies.

Case 8. $x_2, x_5 \in K$. This implies $x_5 \in L$, implying $x_3 \in K$ or $x_4 \in K$. In the first subcase, 6 applies. In the second, 7 applies.

Case 9. $x_2, x_6 \in K$. This implies $x_2 \in L$, implying $x_1 \in K$ or $x_5 \in K$. In the first subcase, 1 applies. In the second, 8 applies.

Case 10. $x_3, x_4 \in K$. This implies $x_2 \in L$, implying $x_1 \in K$ or $x_5 \in K$. In the first subcase, 2 applies. In the second, we have $x_3 \in L$, implying $x_1 \in K$, and 2 applies again.

Case 11. $x_3, x_5 \in K$. This implies $x_3 \in L$, implying $x_1 \in K$ or $x_2 \in K$. Apply 2 or 11.

Case 12. $x_3, x_6 \in K$. This implies $x_2 \in L$, implying $x_1 \in K$ or $x_5 \in K$. Apply 2 or 11.

Case 13. $x_4, x_5 \in K$. This implies $x_3 \in L$, implying $x_1 \in K$ or $x_2 \in K$. Apply 3 or 7.

Case 14. $x_4, x_6 \in K$. This implies $x_2 \in L$, implying $x_1 \in K$ or $x_5 \in K$. Apply 3 or 13.

Case 15. $x_5, x_6 \in K$. This implies $x_4 \in L$, implying $x_2 \in K$ or $x_3 \in K$. Apply 8 or 11.

Therefore, $K = L = X$, and we conclude that $X \times X$ is the unique weak saddle. (The same logic shows that $\mathrm{MC}(M) = X$.) Since $\frac{1}{2}a_1 + \frac{1}{2}a_4 >_{1,2,3,4,5} a_6$, it follows that $\{x_1, x_2, x_3, x_4, x_5\} \times \{x_1, x_2, x_3, x_4, x_5\}$ is we WMGSP. The same logic as before shows that $\{x_1, x_2, x_3, x_4, x_5\} \times \{x_1, x_2, x_3, x_4, x_5\}$ is the unique weak mixed saddle.

A.3 Example 3

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	0	-1	0	0	0	0	0	1
x_2	1	0	0	0	0	0	0	0
x_3	0	0	0	0	0	-1	1	1
x_4	0	0	0	0	1	1	-1	1
x_5	0	0	0	-1	0	1	0	0
x_6	0	0	1	-1	-1	0	0	0
x_7	0	0	-1	1	0	0	0	-1
x_8	-1	0	-1	-1	0	0	1	0

Here, $\{x_1, x_2\} \times \{x_1, x_2\}$ is a weak saddle, and $\{x_3, x_4, x_5, x_6, x_7, x_8\} \times \{x_3, x_4, x_5, x_6, x_7, x_8\}$ is a WMGSP. It therefore includes a weak mixed saddle disjoint from the weak saddle mentioned above. We claim that the weak saddle is unique. Suppose $K \times L$ is a distinct WGSP. Without loss of generality, we may assume $L \cap \{x_3, x_4, x_5, x_6, x_7, x_8\} \neq \emptyset$. We will show that $\{x_3, x_4, x_5, x_6, x_7, x_8\} \subseteq K \cap L$. We consider six cases.

Case 1. $x_3 \in L$. This implies $x_6 \in K$, giving us two subcases.

1.1. $x_4 \in L \Rightarrow x_7 \in K \Rightarrow x_5, x_8 \in L \Rightarrow x_4 \in K \Rightarrow x_7 \in L \Rightarrow x_3 \in K \Rightarrow x_6 \in L \Rightarrow x_5, x_8 \in K$.

1.2. $x_5 \in L \Rightarrow x_4 \in K \Rightarrow x_7 \in L \Rightarrow x_3 \in K \Rightarrow x_6 \in L \Rightarrow x_5, x_8 \in K \Rightarrow x_4 \in L \Rightarrow x_7 \in K \Rightarrow x_8 \in L$.

Case 2. $x_4 \in L$. This implies $x_7 \in K$, giving us two subcases.

2.1. $x_3 \in L$. Apply 1.

2.1. $x_8 \in L$. We have two subcases. If $x_3 \in K$ then, by 1 and symmetry, $K = L = X$. If $x_4 \in K$ then $x_7 \in L$, implying $x_3 \in K$. Apply 1 and symmetry.

Case 3. $x_5 \in L$. Then $x_4 \in K$. Apply 2 and symmetry.

Case 4. $x_6 \in L$. There are two subcases. If $x_4 \in K$ then apply 2 and symmetry. If $x_5 \in K$, we apply 3 and symmetry.

Case 5. $x_7 \in L$. There are two subcases.

5.1. $x_3 \in K$. Apply 1 and symmetry.

5.2. $x_8 \in K$. There are three subcases. If $x_3 \in L$ then apply 1. If $x_4 \in L$ then apply 2. If $x_1 \in L$ then $x_3 \in K$, and we apply 1 and symmetry.

Case 6. $x_8 \in L$. There are three subcases. If $x_3 \in K$, apply 1 and symmetry. If $x_4 \in K$, apply 2 and symmetry. If $x_1 \in K$ then $x_2 \in L$, and then either $x_3 \in K$ or $x_4 \in K$. Apply 1 or 2 and symmetry.

Since $\{x_3, x_4, x_5, x_6, x_7, x_8\} \subseteq K \cap L$, we must have $x_1 \in K \cap L$, and then we must have $x_2 \in K \cap L$. Thus, we have shown that $K = L = X$, implying that the unique minimal WGSP is indeed $\{x_1, x_2\} \times \{x_1, x_2\}$.

A.4 Example 4

	x_1	x_2	x_3	x_4
x_1	0	1	0	0
x_2	-1	0	-1	-1
x_3	0	1	0	1
x_4	0	1	-1	0

Here, MC(M) =UC(M) =GOCHA(M) $= \{x_1, x_3\}$, and $\{x_3, x_4\} \times \{x_3, x_4\}$ is a weak saddle and a weak mixed saddle.

A.5 Example 5

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	0	0	0	0	0	0	0	1
x_2	0	0	1	1	-1	-1	1	1
x_3	0	-1	0	1	1	-1	-1	1
x_4	0	-1	-1	0	1	1	1	1
x_5	0	1	-1	-1	0	1	-1	1
x_6	0	1	1	-1	-1	0	1	1
x_7	0	-1	1	-1	1	-1	0	-1
x_8	-1	-1	-1	-1	-1	-1	1	0

Here, GOCHA$(M) = \{x_1\}$, and $\{x_2, x_3, x_4, x_5, x_6, x_7\} \times \{x_2, x_3, x_4, x_5, x_6, x_7\}$ is a WMGSP, because

$$\begin{aligned} \frac{1}{5}a_2 + \frac{1}{5}a_3 + \frac{1}{5}a_4 + \frac{1}{5}a_5 + \frac{1}{5}a_6 &>_{\{x_2,x_3,x_4,x_5,x_6,x_7\}} a_1 \\ a_6 &>_{\{x_2,x_3,x_4,x_5,x_6,x_7\}} a_8. \end{aligned}$$

Thus, there is a weak mixed saddle disjoint from the GOCHA set. We claim that any WMGSP, say $K \times L$, with $x_1 \in K$ or $x_1 \in L$ must include $\{x_2, x_3, x_4, x_5, x_6, x_7\}$. Without loss of generality, we suppose $x_1 \in L$. Of course, L must contain another alternative. We consider seven cases.

Case 1. $x_1, x_2 \in L$. There are two subcases.

1.1. $x_5 \in K$. There are three subcases.

1.1.1. $x_3 \in L \Rightarrow x_6 \in K \Rightarrow x_4 \in L \Rightarrow x_2 \in K \Rightarrow x_5 \in L \Rightarrow x_3, x_7 \in K \Rightarrow x_6, x_7 \in L \Rightarrow x_4 \in K$.

1.1.2. $x_4 \in L \Rightarrow x_2 \in K \Rightarrow x_5 \in L \Rightarrow x_3 \in K \Rightarrow x_6, x_7 \in L \Rightarrow x_4, x_6 \in K \Rightarrow x_3 \in L \Rightarrow x_7 \in K$.

1.1.3. $x_7 \in L \Rightarrow x_6 \in K \Rightarrow x_4 \in L$. Apply 1.1.2.

1.2. $x_6 \in K$. There are two subcases.

1.2.1. $x_5 \in L \Rightarrow x_5 \in K$. Apply 1.1.

1.2.2. $x_4 \in L \Rightarrow x_2 \in K \Rightarrow x_5 \in L$. Apply 1.2.1.

Case 2. $x_1, x_3 \in L$. There are three subcases.

2.1. $x_2 \in K$. There are two subcases.

2.1.1. $x_5 \in L \Rightarrow x_7 \in K \Rightarrow x_6 \in L \Rightarrow x_4, x_6 \in K \Rightarrow x_4 \in L \Rightarrow x_3 \in K \Rightarrow x_2 \in L$. Apply 1.

2.1.2. $x_6 \in L \Rightarrow x_6 \in K \Rightarrow x_5 \in L$. Apply 2.1.1.

2.2. $x_6 \in K$. There are two subcases.

2.2.1. $x_4 \in L \Rightarrow x_7 \in K \Rightarrow x_4 \in L \Rightarrow x_2, x_3 \in K \Rightarrow x_6 \in L \Rightarrow x_4 \in K \Rightarrow x_2 \in L$. Apply 1.

2.2.2. $x_5 \in L \Rightarrow x_7 \in K \Rightarrow x_4 \in L \Rightarrow x_2, x_3 \in K \Rightarrow x_6 \in L \Rightarrow x_4 \in K \Rightarrow x_2 \in L$. Apply 1.

2.3. $x_7 \in K$. There are three subcases.

2.3.1. $x_2 \in L$. Apply 1.

2.3.2. $x_4 \in L \Rightarrow x_2 \in K$. Apply 2.1.

2.3.3. $x_6 \in L \Rightarrow x_6 \in K$. Apply 2.2.

Case 3. $x_1, x_4 \in L$. There are two subcases.

3.1. $x_2 \in K$. There are two subcases.

3.1.1. $x_5 \in L \Rightarrow x_3 \in K \Rightarrow x_6 \in L \Rightarrow x_4 \in K \Rightarrow x_2 \in L$. Apply 1.

3.1.2. $x_6 \in L \Rightarrow x_4 \in K \Rightarrow x_2 \in L$. Apply 1.

3.2. $x_3 \in K$. There are three subcases.

3.2.1. $x_2 \in L$. Apply 1.

3.2.2. $x_6 \in L \Rightarrow x_4 \in K \Rightarrow x_2 \in L$. Apply 1.

3.2.3. $x_7 \in L \Rightarrow x_2 \in K$. Apply 3.1.

Case 4. $x_1, x_5 \in L$. There are three subcases.

4.1. $x_3 \in K$. There are three subcases.

4.1.1. $x_2 \in L$. Apply 1.

4.1.2. $x_6 \in L \Rightarrow x_4 \in K \Rightarrow x_2 \in L$. Apply 1.

4.1.3. $x_7 \in L \Rightarrow x_4 \in K \Rightarrow x_2 \in L$. Apply 1.

4.2. $x_4 \in K$. There are two subcases. If $x_2 \in L$, apply 1. If $x_3 \in L$, apply 2.

4.3. $x_7 \in K$. There are three subcases. If $x_2 \in L$, apply 1. If $x_4 \in L$, apply 3. If $x_6 \in L$ then $x_4 \in K$, implying $x_2 \in L$. Apply 1.

Case 5. $x_1, x_6 \in L$. There are two subcases.

5.1. $x_4 \in K$. There are two subcases. If $x_2 \in L$, apply 1. If $x_3 \in L$, apply 2.

5.2. $x_5 \in K$. There are three subcases. If $x_3 \in L$ apply 2. If $x_4 \in L$, apply 3. If $x_7 \in L$ then $x_4 \in K$, implying $x_3 \in L$. Apply 2.

Case 6. $x_1, x_7 \in L$. There are four subcases.

6.1. $x_2 \in K$ implies $x_5 \in L$ or $x_6 \in L$. Apply 4 or 5.

6.2. $x_4 \in K$ implies $x_2 \in L$ or $x_3 \in L$. Apply 1 or 2.

6.3. $x_6 \in K$ implies $x_4 \in L$ or $x_5 \in L$. Apply 3 or 4.

6.4. $x_8 \in K$. Then L must contain at least one other point, and the appropriate case applies.

Case 7. $x_1, x_8 \in L$. L must contain at least one other point, and the appropriate subcase applies.

Thus, there is no weak mixed saddle $K \times L$ such that $x_1 \in K$ or $x_1 \in L$.

A.6 Example 6

	x_1	x_2	x_3	x_4	x_5
x_1	0	-1	1	0	0
x_2	1	0	0	0	1
x_3	-1	0	0	-1	1
x_4	0	0	1	0	1
x_5	0	-1	-1	-1	0

Here, $S(M) = MS(M) = R(M) = \{x_1, x_2, x_3, x_4\}$, $X^* = X$, and $\{x_3, x_4, x_5\} \times \{x_3, x_4, x_5\}$ is a weak saddle and a weak mixed saddle.

A.7 Example 7

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	0	1	-1	-1	-1	1	0
x_2	-1	0	1	1	-1	-1	0
x_3	1	-1	0	-1	1	-1	0
x_4	1	-1	1	0	0	0	1
x_5	1	1	-1	0	0	0	1
x_6	-1	1	1	0	0	0	1
x_7	0	0	0	-1	-1	-1	0

Here, GOCHA(M) $= R(M) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Since $\{x_1, x_2, x_3, x_4, x_5, x_6\} \times \{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a MGSP, Theorem 8 implies that MS(M) $= \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Furthermore, GETCHA($M$) =UC($M$) $= X$. We claim that $MC(M) = X$ and that $X \times X$ is the unique weak saddle of the M-game. We first establish the latter claim in several steps. Let $K \times L$ be an arbitrary WGSP, and note that $|L| > 1$.

Step 1: If $x_1, x_2, x_3 \in L$ then $K = L = X$. It clear that $x_4 \in K$, for no other alternative can weakly dominate it over $\{x_1, x_2, x_3\}$. Similarly, $x_5, x_6 \in K$. Given this, we must have $\{x_4, x_5, x_6\} \cap L \neq \emptyset$. There are three cases.

Case 1. $x_4 \in L$. Then $x_2 \in K$, giving us two subcases.

1.1 $x_5 \in L$. This implies $x_3 \in K$, implying $x_6 \in L$, implying $x_1 \in K$, implying $x_7 \in K \cap L$. Thus, $K = L = X$.

1.2 $x_6 \in L$. This implies $x_1 \in K$, implying $x_5 \in L$. Then, by 1.1, $K = L = X$.

Case 2. $x_5 \in L$. Then $x_3 \in K$, giving us two subcases.

2.1 $x_4 \in L$. This implies $x_1 \in K$, implying $x_6 \in L$, implying $x_1 \in K$, implying $x_7 \in K \cap L$. Thus, $K = L = X$.

2.2 $x_6 \in L$. This implies $x_1 \in K$, implying $x_4 \in L$. Then, by 2.1, $K = L = X$.

Case 3. $x_6 \in L$. Then $x_1 \in K$, giving us two subcases.

3.1 $x_4 \in L$. This implies $x_2 \in K$, implying $x_5 \in L$, implying $x_3 \in K$, implying $x_7 \in K \cap L$. Thus, $K = L = X$.

3.2 $x_5 \in L$. This implies $x_3 \in K$, implying $x_4 \in L$. Then, by 3.1, $K = L = X$.

Step 2: If L contains any two elements of $\{x_1, x_2, x_3\}$, it contains all three. If $x_1, x_2 \in L$ then $x_5 \in K$, implying $x_3 \in L$. If $x_1, x_3 \in L$ then $x_4 \in K$, implying $x_2 \in L$. The argument for $x_2, x_3 \in L$ is similar.

Step 3: If L contains any two elements of $\{x_4, x_5, x_6\}$, it contains all three. If $x_4, x_5 \in L$ then $x_2, x_3 \in K$, implying $x_6 \in L$. If $x_4, x_6 \in L$ then $x_2, x_1 \in K$, implying $x_5 \in L$. The argument for $x_5, x_6 \in L$ is similar.

Step 4: If $L \cap \{x_1, x_2, x_3\} \neq \emptyset$ and $L \cap \{x_4, x_5, x_6\} \neq \emptyset$, then L contains at least two elements of $\{x_1, x_2, x_3\}$. Of six cases, we focus on three without loss of generality: $x_1, x_4 \in L$, $x_1, x_5 \in L$, and $x_1, x_6 \in L$. In all three cases, either $x_4 \in K$, implying $x_2 \in L$, or $x_5 \in K$, implying $x_3 \in L$.

Step 5: L contains at least one element of $\{x_1, x_2, x_3\}$. If not, it contains at least two elements of $\{x_4, x_5, x_6\}$. From Step 3, it contains all three. Therefore, $K \cap \{x_4, x_5, x_6\} \neq \emptyset$, implying $L \cap \{x_1, x_2, x_3\} \neq \emptyset$.

Combining Steps 1, 2, 4, and 5, we see that $K = L = X$. The same logic used in each step applies equally well to the minimal covering set, implying $\mathrm{MC}(M) = X$.

A.8 Example 8

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	0	-1	1	0	0	1
x_2	1	0	-1	1	0	0
x_3	-1	1	0	0	1	0
x_4	0	-1	0	0	-1	1
x_5	0	0	-1	1	0	-1
x_6	-1	0	0	-1	1	0

Here, we have GOCHA(M) =MS(M) $= \{x_1, x_2, x_3\}$, and MC(M) =UC(M) = GETCHA(M) $= S(M) = R(M) = X^* = X$. To establish that $MC(M) = X$, first note that if $x_1 \in MC(M)$ then $x_2 \in MC(M)$, and then $x_3 \in MC(M)$. From the symmetric situation of these three alternatives, therefore, we see that if any one of them is in the minimal covering set, they all are. Further, it would then follow that $x_4 \in MC(M)$, because this alternative is not covered in any Y containing x_1, x_2, and x_3; and, similarly, $x_5, x_6 \in MC(M)$.

If $x_4 \in MC(M)$ then either $x_2 \in MC(M)$, in which case we have shown that $MC(M) = X$, or $x_5 \in MC(M)$; if $x_5 \in MC(M)$ then either $x_3 \in MC(M)$, in which case $MC(M) = X$, or $x_6 \in MC(M)$; if $x_6 \in MC(M)$ then either $x_1 \in MC(M)$, in which case $MC(M) = X$, or $x_4 \in MC(M)$. But the minimal covering set must contain at least one alternative in addition to x_4, x_5, and x_6. Thus, $MC(M) = X$, as claimed.

A.9 Example 9

	x_1	x_2	x_3	x_4	x_5
x_1	0	0	0	1	1
x_2	0	0	0	0	1
x_3	0	0	0	0	1
x_4	-1	0	0	0	0
x_5	-1	-1	-1	0	0

Here, $S(M) = \{x_1, x_2, x_3, x_4\}$ and GETCHA$(M) = X$.

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