Equilibrium Non-existence in a Model of Representative Democracy

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Abstract

It is well-known that equilibrium existence problems in multidimensional models of two-candidate elections can be solved if voters have preferences for non-policy characteristics of candidates, an approach often termed "probabilistic voting." We show that, if a continuum of voters are partitioned into a finite number of districts, then all equilibria of the electoral game correspond to undominated (or "core") points of the associated weighted majority voting game among the districts. Results from social choice theory can then be applied, with the conclusion that electoral equilibria will typically fail to exist in higher dimensional policy spaces. In the context of the distributive model of politics, equilibria will typically fail to exist when the number of interest groups is high.

1 Introduction

The widely used benchmark for the analysis of two-candidate elections is the model of Downs (1957), in which candidates simultaneously commit to policy platforms on the real line, each voter votes for the preferred of the two candidates, and the candidate with the majority of votes is elected and implements his/her platform. The median voter theorem states that, if candidates are office-motivated and voter preferences are single-peaked, then the unique Nash equilibrium of the electoral game is for both candidates to locate at the median of the voter ideal points. The property of the median driving this result is, of course, the fact that it is unbeaten against any other policy in pairwise majority voting. In multidimensionsal policy spaces, such points are called "core points" and typically fail to exist (cf. Plott, 1967; Rubinstein, 1979; Schofield, 1983; Cox, 1984; Le Breton, 1987; Banks, 1995). As a consequence, electoral equilibria typically fail to exist as well.

One approach to non-existence of equilibria, pursued by Hinich (1977, 1978), Coughlin and Nitzan (1981), Wittman (1983), Calvert (1985), and Lindbeck and Weibull (1987, 1993), among others, is to assume voting is "probabilistic." In one such model, a voter considers the platforms of the two candidates and votes for one as an increasing function of the utility differential offered by that candidate. A common interpretation is that voters have preferences for non-policy characteristics of the candidates, or for unmodeled policy dimensions along which the candidates are viewed as fixed, that are unobserved and which introduce uncertainty about voting behavior. If the electorate is large and modeled as a continuum, then, because aggregate uncertainty "integrates out," the probability that one type of voter will vote for a candidate can be interpreted as the proportion of such voters, and the assumption of incomplete information becomes unnecessary. Furthermore, this allows us to replace the usual assumption that candidates are expected plurality maximizers with the more intuitive objective of maximizing probability of winning.

It is known that, under appropriate concavity conditions, there exist pure strategy equilibria in the electoral game with probabilistic voting. Furthermore, when probabilities depend on utility differences, Hinich (1977, 1978) and Lindbeck and Weibull (1987, 1993) have given characterizations of equilibrium platforms in terms of a "utilitarian" welfare function: Hinich (1978) does so in the multidimensional spatial model of politics with quadratic utility functions, and Lindbeck and Weibull (1987, 1993) do so in the distributive model of politics. Banks and Duggan (2000) extend those characterizations to the multidimensional spatial model with general voter utility functions. Section 2 contains a discussion of the conditions on primitives sufficient for concavity of the candidates' payoff functions. When they are satisfied, the results of this literature therefore give us a solution to the problem of equilibrium existence in multidimensional models of majority-rule elections.

In this paper, we explore a probabilistic voting model in which a continuum of voters are partitioned into a finite number of districts, with perhaps unequal weights, and the winner is the candidate who wins a weighted majority of districts. After associating each district with a particular "welfare" function, we show that, at every symmetric equilibrium of the electoral game, the gradients of these welfare functions must satisfy a strong symmetry condition; in the terminology defined below, every symmetric equilibrium platform must be a "local median in all directions" for the weighted majority voting game among the districts. In higher dimensions, as in the distributive politics model with a large number of interest groups, this condition will typically not be met by any policies. And when it is met, it will be vulnerable to small changes in voter utility functions, in probability-of-vote functions, and in the distribution of voters across districts. In lower dimensional settings, the condition may be robust to such perturbations, if there is one district with strictly greater weight than all others. In that case, the only robust symmetric equilibria are critical points of that district's welfare function.[1] If the districts are odd in number and equally weighted, then the necessary condition on district gradients reduces to Plott's (1967) restrictive symmetry condition for majority core points.

We can say more if we impose standard concavity conditions on voter utility functions. First, every symmetric equilibrium is actually in the core of the voting game among the districts. Second, there are *no* asymmetric equilibria,

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[1] For the case of one district with concave voter utility functions, this yields the usual equilibrium characterization in terms of a utilitarian welfare optimum.

so the latter claim applies to all equilibria of the electoral game. Third, if we add appropriate concavity conditions to probability-of-vote functions, every policy in the core of the district voting game forms a symmetric equilibrium. Thus, while the core is often empty in higher dimensions, this gives us an exact characterization of equilibria in one dimension: the unique equilibrium of the electoral game has both candidates adopting the "weighted median" of the voting game among the districts, defined with respect to the district ideal points. If voter utilities are quadratic, then a district's ideal point is just the mean ideal point of its constituents. So if districts are equally weighted, the unique equilibrium has both candidates at the median of the district means.

We conclude that, while probabilistic voting may offer existence of equilibria in electoral competition within a single district, it does not yield a satisfactory solution to the existence problem more generally. Other approaches to equilibrium existence in multiple dimensions have been explored as well. Duggan and Fey (2000) suppose that candidates are policy-motivated, making profitable deviations for the candidates more difficult: to profit from a deviation, a candidate must not only win, but must win with a preferable platform. While the results for two dimensions are somewhat encouraging, however, that paper finds that, in three or more dimensions, we are back to generic non-existence of equilibria. Duggan and Fey (2001) consider an infinitely repeated version of the basic Downsian model, obtaining equilibrium existence when the discount factors of voters are greater than one half. In that case, however, *every* path of policies over time can be supported by a subgame perfect equilibrium of the repeated game. A feature common to the model of this paper and to these other models is the assumption that candidates can commit to policies prior to the election. That simplification is useful in one dimension, of course, but appears quite costly in multiple dimensions.

The paper is organized as follows. In Section 2, we present the model. In Section 3, we give our results. And in Section 4, we discuss the extension of our results to a model of probabilistic voting based on utility ratios rather than utility differences, as above.

2 The Model

We model an election with two candidates, A and B, who simultaneously choose policy platforms, x_A and x_B, before an election. Let $C \in \{A, B\}$ denote an arbitrary candidate. The policy space, X, is a convex subset of $\mathbb{R}^m$. Voters are partitioned into a finite number $n \geq 1$ of districts, where D denotes the set of districts, and are described by a measurable space T of types. Each type t of voter has policy preferences represented by a differentiable utility function $u_t \colon X \to \mathbb{R}$, where $u_t(x)$ is measurable in t for each x. The distribution of types in district d is given by the probability measure μ_d, which is assumed to be non-atomic. Thus, we consider a large population of voters partitioned into a relatively small (finite) number of districts. Because the distribution of types is allowed to vary arbitrarily across districts, we allow for the possibility that some types of voters are present in some districts but not in others.

As a special case of this environment, we obtain the one-dimensional spatial model of politics by setting $m = 1$ and each u_t single-peaked. The multidimensional spatial model is obtained by setting $m \geq 2$ and making each u_t quasi-concave with a unique "ideal point," or utility-maximizing policy. We allow for the distributive model of politics by partitioning the voters into a finite set of interest groups, $G_1, G_2, \ldots, G_k$, and setting X equal to the unit simplex in $\mathbb{R}^k$. Because our results concern interior policies, however, we reformulate the distributive politics model so that the set of policies is

$$X = \left\{ (x_1, x_2, \ldots, x_{k-1}) \mid \text{ each } x_j \geq 0 \text{ and } \sum_{j=1}^{k-1} x_j \leq 1 \right\}.$$

Here, x_j is the amount of resource allocated to group G_j for $j < k$, and group G_k is implicitly allocated $1 - \sum_{j=1}^{k-1} x_j$. Letting $j(t)$ index the group to which type t voters belong, $j(\cdot)$ being a measurable function, we specify

$$u_t(x) = v_t(x_{j(t)})$$

for $j(t) < k$, where $v_t \colon [0, 1] \to \mathbb{R}$ is a strictly increasing function, and

$$u_t(x) = v_t(1 - \sum_{j=1}^{k-1} x_j)$$

for $j(t) = k$. Here, interest groups may cut across districts arbitrarily. We require all voters of the same type to belong to the same interest group in this formulation, but that requirement is without loss of generality, because we allow $v_t = v_{t'}$ for different types of voters.

Many of our results use the assumption that each u_t is concave, which is satisfied in the usual formulations of these models. For one result, however, we assume the condition of *aggregate strict concavity,* which is the requirement that $\int u_t \, d\mu_d$ is strictly concave for each district. This is weaker than directly assuming each u_t is strictly concave, a condition typically satisfied in the spatial model but *not* in the distributive model: there, we will always have $u_t(\alpha x + (1-\alpha)y)$ constant in α if $x_{j(t)} = y_{j(t)}$. Aggregate strict concavity *is* satisfied in the distributive model if each v_t is strictly concave and interest groups are *dispersed,* i.e.,

$$\mu_d(\{t \in T \mid j(t) = j\}) \quad > \quad 0$$

for all districts d and groups j. In words, this means that all interest groups are represented in all districts. Aggregate strict concavity is also satisfied in exchange economies under similar conditions.

We assume that voting is probabilistic, and that the probability the type t voters vote for a candidate depends on the difference in utilities offered by the candidates' platforms. Specifically, we assume that, for each $t \in T$, there exists a function $P_t^A \colon \mathbb{R} \to \mathbb{R}$ such that the probability the type t voters vote for A is

$$P_t^A(u_t(x_A) - u_t(x_B)),$$

and we assume the type t voters vote for B with probability $P_t^B = 1 - P_t^A$. Because we posit a continuum of types in each district, we may also view P_t^C as the *proportion* of type t voters who vote for candidate C. We assume that each P_t^C is differentiable, that each P_t^A is non-decreasing, that each P_t^B is non-increasing, and that each $P_t^C(0) = 1/2$, so that the voters treat the candidates symmetrically when indifferent. We use DP_t^C to represent the derivative of P_t^C. This model is referred to as the "utility difference" model in Banks and Duggan (2000) and was introduced by Hinich (1977) and Lindbeck and Weibull (1987, 1993).

This model of probabilistic voting is often interpreted in terms of voter preferences for non-policy characteristics of the candidates, as follows. Suppose each type t voter derives utility from the policy implemented by a candidate and some utility α_t from candidate A, due to personal attributes, unmodeled policy dimensions along which A's position is fixed, etc., and let β_t be the utility from candidate B. Assuming voters are sincere, the type t voters vote for A if

$$u_t(x_A) + \alpha_t \quad > \quad u_t(x_B) + \beta_t$$

and for B if the inequality is reversed. Since β_t may be positive or negative, we can equivalently set $\alpha_t = 0$ and let the type t voters vote for A if

$$\beta_t \quad < \quad u_t(x_A) - u_t(x_B).$$

Here, we may refer to β_t as a "bias" in favor of candidate B. Assume the candidates know the policy utility functions of the voters but not the bias β_t for any type, which is modeled as a random variable with distribution F_t from their point of view. Assuming F_t is continuous, the probability A gets the votes of type t voters is

$$F_t(u_t(x_A) - u_t(x_B)).$$

If F_t is differentiable and $F_t(0) = 1/2$, then this model, called the "additive bias" model in Banks and Duggan (2000), is equivalent to the utility difference model with $P_t^A = F_t$.

Each district d is given a non-negative weight ω_d, normalized so that $\sum_D \omega_d = 1$. We assume these weights are *strong,* meaning that the districts cannot be partitioned into two disjoint collections with equal weight: there do not exist D' and D'' such that $D' \cap D'' = \emptyset$, $D = D' \cup D''$, and $\sum_{D'} \omega_d = \sum_{D''} \omega_d$. When districts have equal weight, this reduces to the assumption that n is odd. Because μ_d is non-atomic, the proportion of votes in district d won by candidate A is

$$\Pi_d^A(x_A, x_B) \quad = \quad \int P_t^A(u_t(x_A) - u_t(x_B))\, \mu_d(dt),$$

and the proportion won by B is $\Pi_d^B(x_A, x_B) = 1 - \Pi_d^A(x_A, x_B)$. We loosely refer to Π_d^C as the *district payoff function* for candidate C. These functions are differentiable, and we use the notation

$$\nabla_{x_C} \Pi_d^C(x_A, x_B)$$

to express the vector of partial derivatives with respect to the policy dimensions of candidate C's own platform. A candidate C wins a majority of votes in district d if

$$\Pi_d^C(x_A, x_B) \quad > \quad \frac{1}{2},$$

and the candidates tie if $\Pi_d^A(x_A, x_B) = \Pi_d^B(x_A, x_B) = 1/2$. In that case, the winner in district d is determined by the toss of a fair coin. We assume that the winner of the election is the candidate who wins in the set of districts with the greatest weight, e.g., A wins if

$$\sum_{d \in D_A} \omega_d \quad > \quad \frac{1}{2},$$

where D_A consists of the districts won by A, after the resolution of coin tosses, if any. (No ties are possible after ties with districts are broken, by the assumption of strong weights.)

We assume that the candidates are probability of winning maximizers, so (x_A^*, x_B^*) is an *equilibrium* of the electoral game if there is no candidate C and platform x_C such that C's probability of winning would increase after a unilateral deviation to x_C. The electoral game is clearly constant sum and, by our assumption that $P_t^C(0) = 1/2$ for all t, its value is one half. This allows us to restate the definition of equilibrium as follows. Letting

$$D_C(x_A, x_B) \quad = \quad \left\{ d \in D \mid \Pi_d^C(x_A, x_B) > \frac{1}{2} \right\}$$

be the districts in which C receives a majority of votes, (x_A^*, x_B^*) is an equilibrium if, for all $y \in X$,

$$\sum_{d \in D_A(\;, x_B^*} \omega_d \qquad \sum_{d\;\; D_B\; y,\;\, {}_B^*} \omega_d \qquad \text{nd} \qquad {}_{d\in\;\; B(\;{}_A^*\, y)} \quad {}_d \geq \quad {}_{d\in\;\; A(\;{}_A^*\, y)} \quad {}_d.$$

We say an equilibrium is *symmetric* if $x^* = x_B$.

For one result, below, we will assume the district payoff functions are concave in a candidate's own platform, i.e., $\Pi_d^A(x_A, x_B)$ is concave in x_A for each x_B, and likewise for B. Since each P_t^A is increasing, we see that

$$P_t^A(u_t(x_A) - u_t(x_B))$$

is concave in x_A if each P_t^A and u_t are concave. Since Π_d^A is an integral of concave functions, it will also be concave in x_A. For candidate B, we also want each P_t^B and u_t to be concave.[2] Since $P_t^B = 1 - P_t^A$, however, this implies that both probability-of-vote functions are linear. In terms of the additive bias model, this means that the bias for B is uniformly distributed over a suitably large interval, a restrictive assumption. But concavity of the probability-of-vote functions is only being used as a sufficient condition here, and it can be weakened if the voter utility functions are correspondingly "more" concave. Suppose, for example, that the probability-of-vote functions are logistic,

$$P_t^A(\Delta) \;=\; \frac{e^{\Delta}}{1+e^{\Delta}},$$

and so not concave, and that $e^{u_t(x)}$ is concave on X. Then

$$P_t^A(u_t(x_A) - u_t(x_B)) \;=\; \frac{e^{u_t(x_A)}}{e^{u_t(x_B)} + e^{u_t(x_A)}}$$

is concave in x_A and convex in x_B, as required. Thus, while concavity of district payoff functions is implied by linear probability-of-vote functions, it does not imply that condition.

3 Results

For each district d, we define a "welfare" function, $W_d\colon X \to \mathbb{R}$, as

$$W_d(x) \;=\; \int DP_t^A(0) u_t(x)\, \mu_d(dt),$$

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[2] Technically, we only need P_t^A and P_t^B to be concave over the range of possible utility differences for type t voters.

and note that W_d is differentiable. Given these welfare functions, it will be useful to consider the weighted majority voting game among the districts, with each district d viewed as a unitary voter with utility function W_d and weight ω_d. The *core* of the district voting game, denoted K, consists of any policy x such that no weighted majority of districts strictly prefers another policy y. To define this set formally, let

$$\mathcal{D} \;=\; \left\{ G \subseteq D \mid \sum_{d \in G} \omega_d > \frac{1}{2} \right\}$$

consist of the collections districts containing more than half the total weight of the districts. Then

$$K \;=\; \left\{ x \in X \mid \begin{array}{l} \text{there do not exist } y \in X \text{ and } G \in \mathcal{D} \text{ such} \\ \text{that, for all } d \in G, W_d(y) > W_d(x) \end{array} \right\}.$$

If $m = 1$ and each u_t is concave, so $X \subseteq \mathbb{R}$ and each W_d is concave, then the core consists of the "weighted medians." If X is compact and aggregate strict concavity holds, then each W_d has a unique maximizer, say x^d, and, because the district weights are strong, there is then exactly one weighted median, denoted x^w. This is defined by the condition

$$\sum_{d : x^d < x^w} \omega_d \;=\; \sum_{d : x^w < x^d} \omega_d.$$

When the districts have equal weights, this is just the median of the district ideal points.

We say x is a *local median in all directions* if, for all $G \in \mathcal{D}$, the zero vector is in the convex hull of the gradients of districts in G, i.e.,

$$0 \in \text{conv}\{\nabla W_d(x) \mid d \in G\}.$$

It is well-known that, if an interior policy is a core point, then it must be a local median in all directions. Indeed, if this did not hold at x^* for some $G \in \mathcal{D}$, then we could strongly separate the zero vector from the convex hull by a hyperplane with normal, say, z. Then moving slightly in the z direction from x^*, we would have a policy preferred to x^* by a weighted majority of districts. Thus, x^* must be "centrally located" with respect to the district gradients. Conversely, if each W_d is concave, then every local median in all directions is in the core (cf. Austen-Smith and Banks, 1999, Theorem 5.6). Our first result characterizes the symmetric equilibria of the electoral game in terms of this gradient condition.

Proposition 1 *If* (x^*, x^*) *is an interior symmetric equilibrium, then* x^* *is a local median in all directions.*

Proof: Consider any symmetric equilibrium (x^*, x^*) with x^* in the interior of X, and note that

$$\Pi_d^A(x^*, x^*) \ = \ \Pi_d^B(x^*, x^*) \ = \ \frac{1}{2}$$

for all districts. If the gradient condition is violated, then

$$0 \notin \mathrm{conv}\{\nabla W_d(x^*) \mid d \in G\}.$$

for some $G \in \mathcal{D}$. By the separating hyperplane theorem, we can find $z \in \mathbb{R}^m$ be such that $z \cdot x > 0$ for all x in the convex hull of $\{\nabla W_d(x^*) \mid d \in G\}$. Using

$$\nabla_{x_A} \Pi_d^A(x^*, x^*) \quad = \quad \nabla W_d(x^*)$$

for every district, this implies, in particular, that

$$\nabla_{x_A} \Pi_d^A(x^*, x^*) \cdot z \quad > \quad 0$$

for all $d \in G$. For $\epsilon > 0$, let $z_\epsilon = x^* + \epsilon z$. Choosing ϵ small enough that $z_\epsilon \in X$ and

$$\Pi_d^A(z_\epsilon, x^*) \ > \ \Pi_d^A(x^*, x^*) \ = \ \frac{1}{2}$$

for all $d \in G$, candidate A can win with probability one by deviating to z_ϵ, a contradiction. ∎

Proposition 1 establishes a necessary condition for interior symmetric equilibria, one that does not use concavity of voter utility functions or candidate payoff functions. In the context of the spatial model of politics, it is known that this condition is quite restrictive in higher dimensions: arbitrarily small perturbations of the district welfare functions, either through voter utilities or through probability-of-vote functions or through district type distributions, can lead to the non-existence of a local median in all directions (cf. Austen-Smith and Banks, 1999, Theorem 6.1). Thus, in higher dimensions, the existence of an interior symmetric equilibrium is razor's edge.

In the distributive model of politics, the meaning of a local median in all directions is not entirely transparent, as the definition is not written in terms of the primitives of the model. Given a district d, let $MV_d(x)$ denote the vector

$$\Bigg(\int_{j^{-1}(1)} DP_t^A(0)v_t'(x_1)\,d\mu_d, \ldots, \int_{j^{-1}(k-1)} DP_t^A(0)v_t'(x_{k-1})\,d\mu_d,$$
$$\int_{j^{-1}(k)} DP_t^A(0)v_t'(1-\sum_{h\neq k} x_h)\,d\mu_d\Bigg)$$

of marginal impacts of the different interest groups at x. Let

$$H \;\; = \;\; \{(\alpha_1, \ldots, \alpha_{k-1}, -\sum_{h\neq k}\alpha_h) \mid \alpha_1, \ldots, \alpha_{k-1} \in \mathbb{R}\}$$

denote the subspace of $\mathbb{R}^k$ orthogonal to the vector $(1, 1, \ldots, 1)$, and let

$$\mathrm{proj}_H MV_d(x)$$

denote the projection of the vector of marginal impacts onto H. Note that x is a local median in all directions if and only if there do not exist $G \in \mathcal{D}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}) \in \mathbb{R}^{k-1}$ such that, for all $d \in G$,

$$\nabla W_d(x) \cdot (\alpha_1, \ldots, \alpha_{k-1}) \;\; > \;\; 0.$$

This holds if and only if there do not exist $G \in \mathcal{D}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, -\sum_{h\neq k}\alpha_h) \in H$ such that, for all $d \in G$,

$$MV_d(x) \cdot (\alpha_1, \ldots, \alpha_{k-1}, -\sum_{h\neq k}\alpha_h) \;\; > \;\; 0.$$

And this is equivalent to

$$0 \in \mathrm{conv}\{\mathrm{proj}_H MV_d(x) \mid d \in G\}$$

for all $G \in \mathcal{D}$. That is, x is a local median in all directions if and only if it is a local median in all directions with respect to the projections of the marginal impact vectors onto the subspace H. Results on non-existence of local medians in all directions, and therefore interior symmetric equilibria, then apply to the distributive model when the dimensionality of H is high, i.e., when the number of interest groups is large.

We can say more about the restrictiveness of a local median in all directions by examining a weaker but more intuitive condition. Given two vectors $y, z \in \mathbb{R}^m$, we write $y \propto z$ if the vectors point in the same direction, i.e., $y = \alpha z$ for some $\alpha > 0$. We say the *weighted Plott conditions* hold at x if for all $z \in \mathbb{R}^m$,

$$\sum_{d:\nabla W_d(x) \propto z} \omega_d - \sum_{d:\nabla W_d(x) \propto -z} \omega_d \;<\; \sum_{d:\nabla W_d(x)=0} \omega_d.$$

Roughly, this means that the weight of districts with gradients pointing in any direction cannot diverge too much from the weight of districts with gradients pointing in the opposite direction. If the districts are equally weighted (as in standard majority voting games) and odd in number, and if x is a critical point for no other district, then this condition reduces to the usual condition,

$$|\{d \in D \mid \nabla W_d(x) \propto z\}| \;=\; |\{d \in D \mid \nabla W_d(x) \propto -z\}|,$$

of Plott (1967). In words, for each district with non-zero gradient at x, there must be another district with gradient pointing in exactly the opposite direction, a condition that is generically violated in two or more dimensions. Standard arguments can be used to establish that every local median in all directions satisfies the weighted Plott conditions, with an immediate negative conclusion for equilibrium existence when districts are equally weighted and the number of dimensions is two or more.

More generally, using the weighted Plott conditions, Proposition 1 implies that an interior symmetric equilibrium must be at a critical point of one of the district welfare functions. Assuming concavity, such policies must actually be district ideal points, which are typically finite in number. When $n = 1$, so there is only one district, this implies the usual utilitarian result: the only possible symmetric equilibrium has the candidates locating at a district welfare maximizer. When $m = 1$, so there is only one dimension, the weighted Plott conditions will be satisfied by the weighted median. When $n > 1$ and $m > 1$, the restrictiveness of the weighted Plott conditions may depend on the weights of the districts. Suppose there is no district with strictly greater weight than every other district, i.e., there are at least two districts, say d and d', with highest weight. Then, in two or more dimensions, the weighted Plott conditions will generically be violated: they can be satisfied only if the gradients of d and d' point in exactly opposite directions, or if the gradients of two or more districts point in exactly the same direction.

The weighted Plott conditions, and therefore the existence of equilibria, will be robust to arbitrarily small changes in district welfare functions only if one district has strictly greater weight than all others. In that case, those conditions will be robust only at the critical points of that district's welfare function. If there is only one such critical point, as when the district's welfare function is strictly concave, then there is only one possible symmetric equilibrium that is robust to small perturbations of welfare functions. And if that policy is not a local median in all directions, then there are no equilibria.

All of the foregoing applies to the marginal impact vectors projected to the subspace H in the distributive model: if x is a local median in all directions, then the weighted Plott conditions must hold with respect to these projections onto H. For example, if each v_t is concave, and if there are three equally weighted districts, then we must have $\mathrm{proj}_H MV_d(x) = 0$ for one district (so x is a welfare maximizer for d), and we must have

$$\mathrm{proj}_H MV_{d'}(x) \quad = \quad -\mathrm{proj}_H MV_{d''}(x)$$

for the other two districts. When there are just three or more interest groups, because H is $k-1$-dimensional, this condition is clearly razor's edge: small perturbations of the marginal impact vector of, say, district d', either through voter utilities or probability-of-vote functions or type distributions, can violate the condition.

The next result assumes concavity of voter utility functions to get a sharper result in terms of the core of the district voting game.

Proposition 2 *Assume each u_t is concave. If (x^*, x^*) is an interior symmetric equilibrium, then $x^* \in K$.*

The proof is simple: since each u_t is concave, each W_d is as well; if (x^*, x^*) is an interior symmetric equilibrium, then Proposition 1 implies that x^* is a local median in all directions; since each W_d is concave, $x^* \in K$. The next result addresses the possibility of asymmetric equilibria. In short, under standard assumptions, there are none. Under those conditions, therefore, Propositions 1 and 2 characterize all possible interior equilibria.

Proposition 3 *Assume each $DP_t^C > 0$, each u_t is concave, and aggregate strict concavity holds. If (x_A^*, x_B^*) is an interior equilibrium, then $x_A^* = x_B^*$.*

Proof: Suppose (x_A^*, x_B^*) is an equilibrium but $x_A^* \neq x_B^*$. Since the electoral game is constant sum with a value of one half, each candidate must receive a payoff of one half in equilibrium, i.e., each must win with probability one half. Letting

$$D_A \;=\; D_A(x_A^*, x_B^*) \quad \text{and} \quad D_B \;=\; D_B(x_A^*, x_B^*),$$

it follows that $\sum_{D_A} \omega_d = \sum_{D_B} \omega_d$. Since the district weights are strong, the districts outside $D_A \cup D_B$ have positive weight, say w, and in those districts the parties are tied: $\Pi_d^C(x_A^*, x_B^*) = 1/2$. Let

$$\begin{aligned} D^+ &= \{d \notin D_A \cup D_B \mid \nabla_{x_A} \Pi_d^A(x_A^*, x_B^*) \cdot (x_B^* - x_A^*) > 0\} \\ D^- &= \{d \notin D_A \cup D_B \mid \nabla_{x_A} \Pi_d^A(x_A^*, x_B^*) \cdot (x_B^* - x_A^*) \leq 0\}, \end{aligned}$$

and note that, because the district weights are strong, there are two cases:

$$\sum_{d \in D^+} \omega_d \;>\; \frac{w}{2} \quad \text{or} \quad \sum_{d \in D^-} \omega_d \;>\; \frac{w}{2}.$$

In the first case, for $\epsilon > 0$, let $z_\epsilon = x_A^* + \epsilon(x_B^* - x_A^*)$. For small enough ϵ, we have

$$\Pi_d^A(z_\epsilon, x_B^*) \;>\; \Pi_d^A(x_A^*, x_B^*) \;=\; \frac{1}{2}$$

for all $d \in D^+$, and, by continuity of Π_d^A, we can choose ϵ small enough that

$$\pi_d^A(x_A^*, x_B^*) \;\; > \;\; \frac{1}{2}$$

for all $d \in D_A$. Since

$$\sum_{d \in D_A \cup D^+} \omega_d \;\; > \;\; \frac{1}{2},$$

A wins by deviating to z_ϵ, contradicting the assumption that (x_A^*, x_B^*) is an equilibrium. In the second case, we have

$$\int \left[DP_t^A(u_t(x_A^*) - u_t(x_B^*)) \nabla u_t(x_A^*) \cdot (x_B^* - x_A^*) \right] \mu_d(dt) \;\; \leq \;\; 0$$

for all $d \in D^-$. Note that, by concavity of u_t, we have

$$\nabla u_t(x_A^*) \cdot (x_B^* - x_A^*) \;\; \geq \;\; \nabla u_t(x_B^*) \cdot (x_B^* - x_A^*)$$

for all $t \in T$. By $DP_t^A > 0$, u_t concave, and aggregate strict concavity, we then have

$$\int \left[DP_t^A(u_t(x_A^*) - u_t(x_B^*)) \nabla u_t(x_B^*) \cdot (x_B^* - x_A^*) \right] \mu_d(dt) \;\; < \;\; 0$$

for all $d \in D^-$. This is equivalent to

$$\nabla_{x_B} \Pi_d^B(x_A^*, x_B^*) \cdot (x_B^* - x_A^*) \;\; > \;\; 0$$

for all $d \in D^-$, and the argument from the first case establishes a profitable deviation for B, a contradiction. ∎

Propositions 1-3 give strong necessary conditions for equilibria of the electoral game in which the candidates choose interior policies. The latter qualification is typically met by all equilibria in the one-dimensional and multidimensional spatial model. We can give reasonable conditions under which this is true of the model of distributive politics as well. Suppose, for example, that interest groups are dispersed, that $\lim_{\alpha \to 0} v_t'(\alpha) = \infty$ for all t, and that $DP_t^A > 0$. Then every equilibrium is interior. To see this, note that, because district weights are strong, in every equilibrium there are some districts d such that

$$\Pi_d^A(x_A^*, x_B^*) \;=\; \Pi_d^B(x_A^*, x_B^*) \;=\; \frac{1}{2}.$$

If $x_j = 0$ for some group G_j with $j < k$, or if $\sum_{j=1}^{k-1} x_j = 1$, then one of the candidates could win those districts by redistributing an arbitrarily small amount of the resource to a group with none. By making this amount small enough, the candidate can keep the districts in which it had a majority of votes, allowing the candidate to win the election.

Our last result establishes a sufficient condition for existence of an equilibrium of the electoral game. In the one-dimensional spatial model, it implies that both candidates locating at the weighted median of the district voting game is an equilibrium of the electoral game. Under the conditions of the previous propositions, it is the *unique* equilibrium. In the distributive model, given our analysis of local medians in all directions, the result implies that equilibria can exist, though under precarious conditions.

Proposition 4 *Assume, given x_B, each $\Pi_d^A(\cdot, x_B)$ is concave and has a unique maximizer. Likewise for Π_d^B. If x^* is interior and $x^* \in K$, then (x^*, x^*) is a symmetric equilibrium.*

Proof: Suppose $x^* \in K$ but (x^*, x^*) is not an equilibrium, so some party, say A, has a profitable deviation. Since $\Pi_d^A(x^*, x^*) = 1/2$ for all districts, this means there exists some other $y \in X$ such that, given platform x^* for B, A's probability of winning with y is greater than one half, i.e.,

$$\sum_{d \in D_A(y, x^*)} \omega_d \;>\; \sum_{d \in D_B(y, x^*)} \omega_d.$$

By concavity of A's district payoff functions in x_A, we have

$$\left\{ d \in D \mid \Pi_d^A(y, x^*) > \frac{1}{2} \right\} \;\subseteq\; \left\{ d \in D \mid \nabla_{x_A} \Pi_d^A(x^*, x^*) \cdot (y - x^*) > 0 \right\},$$

and, because $\Pi_d^A(\cdot, x^*)$ is concave and has a unique maximizer,

$$\left\{ d \in D \mid \begin{array}{c} \nabla_{x_A} \Pi_d^A(x^*, x^*) \cdot (y - x^*) < 0 \\ \text{or } \nabla_{x_A} \Pi_d^A(x^*, x^*) = 0 \end{array} \right\} \;\subseteq\; \left\{ d \in D \mid \Pi_d^A(y, x^*) < \frac{1}{2} \right\}.$$

Define

$$
\begin{aligned}
D^+ &= \{d \in D \mid \nabla W_d(x^*) \cdot (y - x^*) > 0\} \\
D^- &= \{d \in D \mid \nabla W_d(x^*) = 0 \text{ or } \nabla W_d(x^*) \cdot (y - x^*) < 0\}.
\end{aligned}
$$

Thus, using $\nabla_{x_A} \Pi_d^A(x^*, x^*) = \nabla W_d(x^*)$, we have

$$\sum_{d \in D^+} \omega_d \; > \; \sum_{d \in D^-} \omega_d.$$

Pick $z \in \mathbb{R}^m$ such that $\nabla W_d(x^*) \cdot z \neq 0$ for all $d \notin D^+ \cup D^-$, and suppose, without loss of generality, that

$$\sum_{d \in D^0 : \nabla W_d(x^*) \cdot z > 0} \omega_d \; \geq \; \sum_{d \in D^0 : \nabla W_d(x^*) \cdot z < 0} \omega_d.$$

Let D^0 consist of all districts outside $D^+ \cup D^-$ with $\nabla W_d(x^*) \cdot z > 0$. For $\epsilon > 0$, let $z_\epsilon = y - x^* + \epsilon z$, and choose ϵ small enough that $\nabla W_d(x^*) \cdot z_\epsilon > 0$ for all $d \in D^+$. Note that

$$\nabla W_d(x^*) \cdot z_\epsilon \; = \; \epsilon \nabla W_d(x^*) \cdot z \; > \; 0$$

for all $d \in D^0$. For $\delta > 0$, let $z_\delta = x^* + \delta z_\epsilon$. Choosing δ small enough, we have $z_\delta \in X$ and $W_d(z_\delta) > W_d(x^*)$ for all $d \in D^+ \cup D^0$, which is a weighted majority of districts, contradicting $x^* \in K$. ∎

4 An Extension

Finally, we comment on an equivalent formulation of our probabilistic voting model, one introduced by Coughlin and Nitzan (1981), in which the probability the type t voters vote for a candidate depend on the *ratio* of utilities rather than the difference. That is, for each $t \in T$, u_t is positive and there exists a function $P_t^A \colon \mathbb{R}_{++} \to \mathbb{R}$ such that the probability the type t voters vote for A is

$$P_t^A \left(\frac{u_t(x_A)}{u_t(x_B)} \right),$$

and $P_t^B = 1 - P_t^A$. This model can be transformed into a utility difference model by defining new utility and probability-of-vote functions as follows:

$$\begin{aligned} \dot{u}_t(x) &= \ln(u_t(x)) \\ \dot{P}_t^A(\Delta) &= P_t^A(e^{\Delta}). \end{aligned}$$

This associated utility difference model is equivalent in the sense that the candidates' district payoff functions, and therefore the equilibria of the electoral game, are invariant with respect to the above transformation:

$$\dot{P}_t^A(\dot{u}_t(x_A) - \dot{u}_t(x_B)) = P_t^A\left(\frac{u_t(x_A)}{u_t(x_B)}\right).$$

Thus, our results hold with only superficial modifications using the utility ratio model.

If we assume the utility ratio model as a primitive, then Proposition 1 will hold for the associated utility difference model, where the gradient condition is expressed in terms of the district welfare functions

$$W_d(x) = \int D\dot{P}_t^A(0)\dot{u}_t(x)\, d\mu_d.$$

Of course, in terms of the primitive utility functions u_t, this is equivalent to

$$W_d(x) = \int D\dot{P}_t^A(0)\ln(u_t(x))\, d\mu_d,$$

which is known as the "Nash" welfare function. Proposition 2 holds as exactly as before, when stated in terms of the associated utility functions $\dot{u}_t$. As we require concavity in the utility difference model, we then need only log-concavity in terms of the primitive utility functions u_t. Similar remarks hold for Proposition 3.

Finally, Proposition 4 uses the assumption that district payoff functions are concave in a candidate's own strategy. From our discussion in Section 2, this will follow if the associated probability-of-vote functions $\dot{P}_t^A$ are linear and the associated utility functions $\dot{u}_t$ are concave, i.e., the primitive utility functions are log-concave. As we mentioned there, however, we can weaken linearity of the $\dot{P}_t^A$ if we strengthen our concavity assumptions on utility functions. We saw that the associated probability-of-vote functions could have the logistic form,

$$\dot{P}_t^A(\Delta) \quad = \quad \frac{e^{\Delta}}{1+e^{\Delta}},$$

if each $e^{\dot{u}_t(x)}$ were concave in x. We can convert the latter conditions into the utility ratio model as follows. The associated probability-of-vote functions will have the logistic form if

$$P_t^A(R) \quad = \quad \frac{R}{1+R},$$

which is the binary Luce model used by Coughlin and Nitzan (1981). And the associated utility functions will be sufficiently concave if each u_t is concave. Under those conditions, then, the concavity assumption of Proposition 4 holds and the result carries over to the utility ratio model.

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