Collective Choice in Linear Environments and Applications to Bargaining and Social Choice

John Duggan
Department of Political Science
and Department of Economics
University of Rochester
Rochester, NY 14627

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Abstract

This note considers a property, called "limited shared weak preferences" (LSWP), in the context of a model in which agents have linear utility functions defined on a polyhedron of alternatives. As special cases, we capture the distributive politics model with externalities in consumption, and we capture collective choice of lotteries by expected utility-maximizing agents. We give a simple sufficient condition for LSWP, which plays an important role in the analysis of stationary equilibria in spatial bargaining games and in the analysis of covering and dominance in social choice theory.

1 Introduction

A classical model of collective choice in political science and economics is the "divide-the-dollar" model, in which a set of agents must allocate a fixed amount of some resource among themselves and in which each agent is only concerned with maximizing his/her own consumption of the resource. The model plays an important role in social choice theory, where it is easily shown that the majority core – the allocations which cannot be improved on for a majority of agents – is empty. In contrast, a related solution imported from the theory of tournaments, called the "uncovered set," consists of all (or nearly all) allocations.[1] As well, the model is central in bargaining theory. It is, for example, one of the main building blocks in the model of Baron and Ferejohn (1989), who first applied bargaining theory to the analysis of legislative decision making.

Work on the uncovered set and on stationary bargaining equilibria has been generalized beyond the divide-the-dollar model. McKelvey (1986) analyzes the uncovered set in the context of the multidimensional spatial model from political science, and Banks, Duggan, and Le Breton (2003) extend the analysis to a general setting, including economic environments. Banks and Duggan (2000, 2003) extend the bargaining analysis of Baron and Ferejohn (1989) also to the general setting. In moving beyond the divide-the-dollar model, a restriction on individual preferences, called "limited shared weak preference" (LSWP), plays a key role in both theories. The condition requires that, if an alternative y is weakly preferred to alternative x by the members of any coalition, then there exist alternatives arbitrarily close to x that are strictly preferred to x by the members of that coalition.

It is straightforward to verify that LSWP is satisfied in the divide-the-dollar model, and Banks and Duggan (1999) establish LSWP in relatively general classes of environments, including exchange economies and including environments with a convex set of public goods and an additively separable private good. In these environments, however, some form of strict convexity or monotonicity plays an important role in verifying LSWP. The extension to the general linear setting presents special technical problems that have not been addressed.

Two environments of special interest are omitted, as a consequence, from the domain of these collective choice theories. The first is the divide-the-dollar model with linear externalities in consumption, as analyzed in Calvert and Dietz (2004). Here, agent i's utility is given by a collection of coefficients, say $u^i_1, \ldots, u^i_n$, one for each agent, where u^i_j/u^i_k (when defined) measures i's fixed marginal rate of substitution for j's consumption relative to k's. In the standard divide-the-dollar model, of course, we have $u^i_i = 1$ and $u^i_j = 0$ for all $j \neq i$. The second is the model in which there is a finite number of "pure" alternatives, and in which the objects of collective choice are lotteries over pure alternatives. Here, an alternative is a vector of probabilities of pure alternatives, and linear utilities are interpreted in terms of von Neumann-Morgenstern expected utility.

[1] See Epstein (1998) and Penn (2002).

The LSWP condition is not satisfied in the general linear setting, where individual preferences are given by gradient vectors: if the gradient vectors of two agents point in opposite directions, for example, then LSWP is violated. This example is non-generic, however, raising the question of the restrictiveness of LSWP in linear environments. In this note, we offer a relatively transparent sufficient condition, called "limited positive linear dependence" (LPLD), for LSWP, and we apply it to the divide-the-dollar model with externalities and to the lottery model. We show that LSWP is generically satisfied in these environments.

We then discuss some implications of these observations for social choice and bargaining theory. Letting $R(x)$ denote the set of alternatives that are weakly majority-preferred to x, Banks, Duggan, and Le Breton (2003) show that LSWP implies $R(\cdot)$, is lower hemi-continuous as a correspondence. A further consequence is that the "undominated set" (a subset of the uncovered set) is non-empty. Also when LSWP is satisfied, Banks and Duggan (2000, 2003) prove general existence and continuity results for stationary bargaining equilibria. We conclude that, generically in the divide-the-dollar model with externalities and in the lottery model, the undominated set is non-empty, and the set of stationary bargaining equilibria is non-empty and varies continuously (in the appropriate sense) in the parameters of the model.

2 Formalities

Let $X \subseteq \Re^d$ consist of the solutions to the finite system of weak inequalities, $p^j \cdot x \geq c_j$, where the index j ranges over the finite set J. Thus, X is closed and convex. Let $u^i \in \Re^d$ denote the gradient of a linear function, where i ranges over the finite set I of n elements. We interpret X as a set of alternatives to be chosen from, I as a set of agents, and u^i as the utility function of agent i. We denote i's weak and strict upper contour sets at x, respectively, by

$$\begin{aligned} R_i(x) &= \{y \in X \mid u^i \cdot y \geq u^i \cdot x\} \\ P_i(x) &= \{y \in X \mid u^i \cdot y > u^i \cdot x\}, \end{aligned}$$

and we let

$$R_C(x) = \bigcap_{i \in C} R_i(x) \quad \text{and} \quad P_C(x) = \bigcap_{i \in C} P_i(x).$$

That is, $R_C(x)$ consists of the policies that every member of C weakly prefers to x, with a similar interpretation of $P_C(x)$. Given $Y \subseteq X$, let $\overline{Y}$ denote the closure of Y.

We say *Limited Shared Weak Preference (LSWP)* holds if, for all $C \subseteq I$ and all $x \in X$,

$$|R_C(x)| > 1 \text{ implies } R_C(x) \subseteq \overline{P_C(x)}.$$

That is, if y (distinct from x) is weakly preferred to x by all members of a group C, then x can be approximated by policies that all members of C strictly prefer to x.

Banks and Duggan (1999) verify that LSWP holds in the classical *divide-the-dollar model* of distributive politics, described as follows. Here, the set of alternatives is the unit simplex in $\Re^n$,

$$X = \left\{ (x_1, \ldots, x_n) \in \Re^n_+ \mid \sum_{i \in I} x_i \leq 1 \right\},$$

representing allocations of a fixed resource across legislative districts, and each agent i's utility is equal to its consumption of the dollar. To capture this model, we let $I = \{1, \ldots, n\}$; let $J = I \cup \{0\}$; let $p^0 = (-1, \ldots, -1)$ and $c_0 = -1$; let $p^j = e^j$, the jth unit vector, and $c_j = 0$ for each $j = 1, \ldots, n$; and let $u^i = e^i$ for each $i = 1, \ldots, n$, i.e., $u^i \cdot x = x_i$. While Banks and Duggan (1999) also allow for non-linear utilities and verify LSWP in a variety of environments, they leave open the case of general linear environments, where the set X may not be the unit simplex and utility vectors may be more general.

The general linear model of this note covers two applications of interest. First, the *divide-the-dollar model with externalities,* examined in Calvert and Dietz (2004), allows for externalities in consumption of money. Here, the set X is again the unit simplex in $\Re^n$, but we allow for arbitrary utility vectors $u^i = (u^i_1, \ldots, u^i_n)$, where u^i_j / u^i_k (when defined) measures i's marginal rate of substitution for j's consumption relative to k's. Second, the *lottery model* views vectors in X as probability distributions over a finite set of "pure" alternatives, $\{a_1, \ldots, a_d\}$. Here, X is the outer face of the unit simplex, i.e.,

$$X = \left\{ (x_1, \ldots, x_d) \in \Re^d_+ \mid \sum_{k=1}^{d} x_k = 1 \right\},$$

and x_k is the probability of alternative a_k. Because this set of alternatives is defined by an equality, we must add a vector $p^{n+1} = (1, \ldots, 1)$ and scalar

$c_{n+1} = 1$ to the above specification. Assuming the agents' preferences over lotteries satisfy the von Neumann-Morgenstern axioms, they admit linear utility representations, given by u^i. Under this interpretation, we then assume then without loss of generality (by a positive affine transformation) that each u^i lies in the outer face of the unit simplex, i.e., $u^i \in X$.

Given any set $Y \subseteq \Re^d$, we denote by lin$[Y]$ the linear hull of (or subspace spanned by) Y. Given a vector $x \in \Re^d$, we write $x = (x_1, \ldots, x_d) > 0$ if $x_j > 0$ for each component j, in which case we say x is "positive." We write $x \geq 0$ if $x_j \geq 0$ for each component, with strict inequality for at least one, in which case x is "semi-positive." We say a set $\{y^1, \ldots, y^m\}$ is *positively linearly dependent* if there exist $a_1, \ldots, a_m \in \Re$ such that

$$\sum_{j=1}^{m} a_j y^j \quad = \quad 0,$$

where $a_j > 0$ for all $j = 1, \ldots, m$. In other words, $\{y^1, \ldots, y^m\}$ is positively linearly dependent if, when we form the matrix M with columns y^j ($j = 1, \ldots, m$), the system of equations $Ma = 0$ has a positive solution.

Finally, we say *Limited Positive Linear Dependence (LPLD)* holds if, for all $C \subseteq I$ and all $D \subseteq J$, if $\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is positively linearly dependent, then

$$\text{lin}[\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}] \quad \supseteq \quad \text{lin}[X - x],$$

where x is any element of X. In words, LPLD holds if, whenever the zero vector can be written as a convex combination of some subset of the u^i and p^j vectors, then that subset spans X, when translated to the origin.

3 Sufficient Conditions

The LPLD condition turns out to be a key sufficient condition for establishing LSWP. Its use is illustrated in Figure 1, where u^1 and u^2 point in opposite directions, creating a violation of LSWP: though there are points $y \in X$ such that $u^1 \cdot y \geq u^1 \cdot x$ and $u^2 \cdot y \geq u^2 \cdot x$, there is no element of X for which these inequalities hold strictly. Similarly, if u^1 and p^j point in opposite directions, there are points $y \in X$ such that $u^1 \cdot y \geq u^1 \cdot x$ and $u^3 \cdot y \geq u^3 \cdot x$, but x maximizes u^i within X.

[Figure 1 here.]

LPLD precludes the situations depicted above, for it requires that X, translated, lie within the line spanned by u^1 and u^2 (or p^j). In this case, there is no $y \in X$ distinct from x such that $u^1 \cdot y \geq u^1 \cdot x$ and $u^2 \cdot y \geq u^2 \cdot x$, fulfilling LSWP. The next lemma generalizes these insights.

Lemma 1 *If LPLD holds, then LSWP holds.*

Proof: Take any $C \subseteq I$, any $x \in X$, and any $y \in R_C(x) \setminus \{x\}$, which implies $y \cdot u^i \geq x \cdot u^i$ for all $i \in C$ and $y \cdot p^j \geq c_j$ for all $j \in J$. Let D consist of the indices $j \in J$ such that $x \cdot p^j = c_j$. Letting $e = |C| + |D|$, form the $d \times e$ matrix M by using u^i ($i \in C$) and p^j ($j \in D$) as columns. If $zM > 0$ has a solution, we are done: define $z_\epsilon = x + \epsilon z$, for arbitrarily small $\epsilon > 0$, to fulfill LSWP. We will see that the remaining case produces a contradiction. If $zM > 0$ has no solution, then the Theorem of the Alternative (Gale, 1960, Theorem 2.9) implies that $Mw = 0$ has a semi-positive solution, say $w = (w_1, \ldots, w_e)$. Let K be the subset of indices $k \in C \cup D$ such that $w_k > 0$, and note that LPLD implies that $\text{lin}[\{m^k \mid k \in K\}] \supseteq \text{lin}[X - x]$. Let K^* index a maximal linearly independent subset of $\{m^k \mid k \in K\}$, so that $y - x \in \text{lin}[\{m^k \mid k \in K^*\}]$. Letting M^* be the submatrix of M obtained by keeping just those columns, we see that $M^* w = 0$ has a positive solution. Thus, by another version of the Theorem of the Alternative (Gale, 1960, Corollary 2), $zM^* \geq 0$ has no solution. If $y \cdot m^k = x \cdot m^k$ for each $k \in K^*$, then linear independence of $\{m^k \mid k \in K^*\}$ implies $y = x$, a contradiction. Since $K^* \subseteq C \cup D$, we cannot have $y \cdot m^k < x \cdot m^k$ for any $k \in K^*$. Therefore, we have $y \cdot m^k \geq x \cdot m^k$ all $k \in K^*$, with strict inequality for at least one $k \in K^*$. This implies $(y - x)M^* \geq 0$, a final contradiction. ∎

To see that LPLD holds in the divide-the-dollar model, suppose that, for some $C \subseteq I$ and $D \subseteq J$, the set $\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is positively linearly dependent. Then it is clear that $C \cup D$ must contain all unit coordinate vectors as well as p^0, i.e., $C \cup D = J$, and then it follows that the linear hull of $\{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is indeed $\Re^n$, fulfilling LPLD.

In fact, LPLD holds in the divide-the-dollar model with externalities under the following generic restriction:

(C1) For all $i \in I$, $u^i \in \Re^n_+ \setminus \{0\}$, and $(1, \ldots, 1)$ cannot be written as a semi-positive combination of any $n - 1$ elements of $\{u^1, \ldots, u^n, p^1, \ldots, p^n\}$.

To see that this condition is sufficient for LPLD, suppose that the zero vector lies in the convex hull of

$$Z \quad = \quad \{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}.$$

Then $0 \in D$ and, by assumption, $|C| + |D| \geq n$. If Z has dimension n, then of course X is contained in its linear hull. Otherwise, it has lower dimension. By Caratheodory's theorem, however, we may select $z^1, \ldots, z^n$ from Z so that 0 lies in the convex hull of $\{z^1, \ldots, z^n\}$, but then necessarily $z^k = (-1, \ldots, -1)$ for some k with positive weight, violating the proposed condition.

A perhaps simpler, yet still generic, sufficient condition is that no subset of $\{u^i \mid i \in I\} \cup \{p^j \mid j \in J\}$ containing n or fewer vectors is linearly dependent. To see sufficiency of this condition, suppose $Z = \{u^i \mid i \in C\} \cup \{p^j \mid j \in D\}$ is positively linearly dependent, and therefore linearly dependent. By assumption, then, $|C| + |D| \geq n + 1$. Furthermore, $\mathrm{lin}[Z]$ has dimension n, for otherwise we could remove an element of Z to arrive at a smaller linearly dependent set. We conclude that $\mathrm{lin}[Z] = \mathrm{lin}[X]$, as required.

Because the set of alternatives in the lottery model is defined by the equality $\sum_{j=1}^{d} x_j = 1$, we need two vectors, $p^0 = (-1, \ldots, -1)$ and $p^{n+1} = (1, \ldots, 1)$, to describe X as the solution to a system of weak inequalities. Since these two vectors are positively linearly dependent, however, LPLD would require that X lies in their linear hull and, therefore, that X is one-dimensional. Thus, we capture the lottery model with d pure alternatives indirectly, by translating X to

$$\hat{X} \quad = \quad \left\{ (x_1, \ldots, x_{d-1}) \in \Re_+^{d-1} \mid \sum_{k=1}^{d-1} x_k \leq 1 \right\},$$

which we define with vectors $\hat{p}^0 = (-1, \ldots, -1)$ and $\hat{p}^j = e^j$ $(j = 1, \ldots, d - 1)$, and by translating u^i to

$$\hat{u}^i \quad = \quad (u_1^i - u_d^i, \ldots, u_{d-1}^i - u_d^i).$$

Given $x \in X$, write $\hat{x} \in \hat{X}$ for the truncation of x, i.e., $\hat{x} = (x_1, \ldots, x_{d-1})$. It is straightforward to check that, for all $x, y \in X$,

$$u^i \cdot x \geq u^i \cdot y \quad \Leftrightarrow \quad \hat{u}^i \cdot \hat{x} \geq \hat{u}^i \cdot \hat{y},$$

so that $(x, u^1, \ldots, u^n) \mapsto (\hat{x}, \hat{u}^i, \ldots, \hat{u}^n)$ is a linear isomorphism that preserves preferences. Thus, if LSWP is satisfied in the new environment, then it is satisfied in the original.

As discussed with respect to the divide-the-dollar model with externalities, the LPLD condition will hold in the new environment under the following generic restriction:

(C2) No subset of $\{\hat{u}^i \mid i \in I\} \cup \{\hat{p}^j \mid j \in J\}$ containing $d-1$ or fewer vectors is linearly dependent.

Mapping back to the original model, we see that LSWP is once again generically satisfied.

4 Implications

Let $\mathcal{D}$ be a collection of non-empty subsets of I such that $(*)$ if $C \in \mathcal{D}$ and $C \subseteq C'$, then $C' \in \mathcal{D}$, and let $\mathcal{B}$ be the dual to this collection, i.e., $C \in \mathcal{B}$ if and only if $I \setminus C \notin \mathcal{D}$. Then define the corresponding strict and weak social preference relations, respectively, by

$$\begin{aligned} P(x) &= \bigcup_{C \in \mathcal{D}} P_C(x) \\ R(x) &= \bigcup_{C \in \mathcal{B}} R_C(x). \end{aligned}$$

Thus, P is irreflexive, R is reflexive, and the relations are dual in the sense that, for all $x, y \in X$, xPy if and only if not yRx. In the linear environments examined above, it is clear that the relation R is closed and P is open. With Proposition 14 of Banks, Duggan, and Le Breton (2003), we also have the following proposition.

Proposition 1 *Assume LPLD holds. Then $R(\cdot)$ is lower hemi-continuous as a correspondence.*

We say x *covers* y, written xCy, if xPy, $P(x) \subseteq P(y)$, and $R(x) \subseteq R(y)$, and we define the *uncovered set,* denoted UC, is the maximal elements of the covering relation, i.e.,

$$UC = \{x \in X \mid \text{ there is no } y \text{ such that } yCx\}.$$

McKelvey (1986) and Banks, Duggan, and Le Breton (2003) show that compactness of X (along with continuity of agents' preferences, which is satisfied in the linear model) is sufficient for the existence of uncovered alternatives. In addition to its properties as a social choice solution, McKelvey (1986) suggests the usefulness of the uncovered set as a bound on mixed strategy equilibrium outcomes of Downsian electoral competition. This conjecture is proved by Banks, Duggan, and Le Breton (2002).

We say x *dominates* y, written xDy, if $P(x) \subseteq P(y)$ and $R(x) \subseteq R(y)$, with at least one inclusion strict. The *undominated set,* denoted UD, is the maximal elements of the dominance relation, i.e.,

$$UD \quad = \quad \{x \in X \mid \text{ there is no } y \text{ such that } yDx\},$$

which forms a subset of the uncovered set. While D is transitive, it does not generally possess desirable continuity properties, so non-emptiness of the undominated set does not follow from standard results. Like the uncovered set, the undominated set also has foundations from non-cooperative game theory: as discussed by McKelvey (1986), the undominated set corresponds to the undominated strategies of office-motivated candidates in the Downsian electoral model. With Proposition 23 of Banks, Duggan, and Le Breton (2003), we have the following result.

Proposition 2 *Assume X is compact and LPLD holds. Then $UD \neq \emptyset$.*

Banks and Duggan (2000, 2003) consider the following infinite-horizon model of bargaining, based on the closed-rule model of Baron and Ferejohn (1989). It is presented here in the context of linear environments. If no alternative has been accepted prior to period t, then (1) fixing recognition probabilities, $\rho_1, \ldots, \rho_n$, agent i is recognized with probability ρ_i; then (2) the selected agent i makes a proposal $p^i \in X$; then (3) every agent $j \in N$ simultaneously votes to either *accept* or *reject* the proposal; then (4) fixing a collection $\mathcal{D} \subseteq 2^N \setminus \{\emptyset\}$ satisfying $(*)$ above, if the group of agents voting for the proposal belongs to $\mathcal{D}$, i.e., if $\{j \in N \mid j \text{ accepts }\} \in \mathcal{D}$, then the proposal p^i is the chosen alternative and bargaining ends with outcome p^i in period t and in every subsequent period. Otherwise, the outcome in period t is the status quo, and steps 1–4 are repeated for period $t+1$.

Each agent i's preferences over outcomes of the bargaining game are described by a von Neumann-Morgenstern linear utility representation $u^i \in$

$\Re^d$ and a discount factor $\delta_i \in [0,1)$ as follows. If alternative x is passed at time t, then agent i's payoff is

$$(1-\delta_i^{t-1})\overline{u}^i + \delta_i^{t-1}(u^i \cdot x),$$

where $\overline{u}^i$ is a fixed status quo payoff. If no alternative is ever passed, then i's payoff is just $\overline{u}^i$. We maintain either of two assumptions.

(A1) There exists $q \in X$ such that, for all $i \in I$, $\overline{u}^i = u^i \cdot q$; and, for all $i, j \in I$, $\delta_i = \delta_j$.

(A2) For all $i \in I$ and all $x \in X$, $\overline{u}^i \leq u^i \cdot x$.

The first assumption formalizes the idea that the status quo could be any alternative under consideration (but adds commonality of the discount factor), while the second formalizes the idea of a "bad" status quo (and allows heterogeneous discount factors).

Complete information of preferences, the structure of the game form, etc., is assumed, and stationary subgame perfect equilibrium is defined in the usual way. In addition, while we allow for proposers to use mixed proposal strategies, we restrict attention to equilibria in which agents use pure voting strategies. In fact, we impose the refinement of stage-game weak dominance: after an alternative, say x, is proposed, an agent i votes for x only if the utility from x weakly exceeds i's "reservation value," given by $(1-\delta_i)\overline{u}^i+\delta_i v_i$, where v_i is i's continuation value from moving bargaining to the next period. This standard refinement eliminates implausible equilibria in which some agents, because their vote is not pivotal, best respond by voting for the worst of two choices. With Theorem 1 of Banks and Duggan (2000) and Theorem 1 of Banks and Duggan (2003), we have the following result.

Proposition 3 *Assume X is compact and LPLD holds. Then there exists a stationary subgame perfect equilibrium in stage-undominated voting strategies.*

In addition, Banks and Duggan (2000, 2003) prove that, if LSWP is satisfied, then the set of stationary equilibrium proposal strategies is upper hemi-continuous in the parameters of the model: recognition probabilities, discount factors, and (in the context of the linear model) in the utility gradients of the agents.[2] Thus, stationary bargaining equilibria possess the type

[2]Here, we endow the space of mixed proposal strategies with the weak* topology. See Banks and Duggan (2000, 2003) for other results on the possibility of delay, etc.

of robustness property expected of equilibrium correspondences: a slight misspecification of the model cannot lead to equilibrium outcomes far from the original equilibrium set.

We conclude that, under (C1) in the divide-the-dollar model with externalities or under (C2) in the lottery model, existing results from social choice and bargaining extend to these two environments. Generically, the undominated set (and therefore the uncovered set) is non-empty, stationary bargaining equilibria exist for the Baron-Ferejohn closed rule bargaining protocol, and the latter vary upper hemi-continuously in the parameters of the bargaining model.

References

[1] J. Banks and J. Duggan (1999) "A Bargaining Model of Collective Choice," Caltech Social Science Working Paper no. 1053.

[2] J. Banks and J. Duggan (2000) "A Bargaining Model of Collective Choice," *American Political Science Review,* 94:73-88.

[3] J. Banks and J. Duggan (2003) "A Bargaining Model of Legislative Policy-Making," mimeo.

[4] J. Banks, J. Duggan, and M. Le Breton (2002) "Bounds for Mixed Strategy Equilibria and the Spatial Model of Elections," *Journal of Economic Theory,* 103:88-105.

[5] J. Banks, J. Duggan, and M. Le Breton (2003) "Social Choice and Electoral Competition in the General Spatial Model," mimeo.

[6] D. Baron and J. Ferejohn (1989) "Bargaining in Legislatures," *American Political Science Review,* 83:1181-1206.

[7] R. Calvert and N. Dietz (2004) "Legislative Coalitions in a Bargaining Model of Externalities," mimeo.

[8] D. Epstein (1998) "Uncovering Some Subtleties of the Uncovered Set: Social Choice Theory and Distributive Politics," *Social Choice and Welfare,* 15:81-93.

[9] D. Gale (1960) *The Theory of Linear Economic Models,* Chicago: University of Chicago Press.

[10] R. McKelvey (1986) "Covering, Dominance, and Institution-free Properties of Social Choice," *American Journal of Political Science,* 30:283-314.

[11] E. Penn (2002) "Alternative Definitions of the Uncovered Set and Their Implications," mimeo.