Electoral Competition with Privately-Informed Candidates*

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Abstract

We consider a model of elections in which two office-motivated candidates receive private signals about the location of the median voter's ideal point prior to taking policy positions. We show that at most one pure strategy equilibrium exists and provide a sharp characterization, if one exists: After receiving a signal, each candidate locates at the median of the distribution of the median voter's location, conditional on the other candidate receiving the *same* signal. It follows that a candidate's position, conditional on his/her signal, is a biased estimate of the true median, with candidate positions tending to the extremes of the policy space. We provide sufficient conditions for the existence of a pure strategy equilibrium. Essentially, the pure strategy equilibrium exists if for each signal, the other candidate is sufficiently likely to receive a signal in the same direction that is at least as extreme. Though the electoral game exhibits discontinuous payoffs for the candidates, we prove that mixed strategy equilibria exist generally, that equilibrium expected payoffs are continuous in the parameters of the model, and that mixed strategy equilibria are upper hemicontinuous. We investigate the robustness of the median voter theorem to private information: Pure strategy equilibria may fail to exist in models close to the Downsian model, but mixed strategy equilibria must, and they will be "close" to the Downsian equilibrium. Finally, we provide bounds on the support of mixed strategy equilibria and restrictions on possible atoms of equilibrium mixed strategies.

1 Introduction

The most familiar and widely-used model of elections in political science and political economy is the classical Downsian model (Hotelling (1929), Downs (1957), Black (1958)). The central result is the median voter theorem: The unique Nash equilibrium in an election between two office-motivated candidates is that both candidates locate at the median voter's ideal point. The logic is simple for symmetric equilibria: If both candidates locate at a position that is not the median voter's, then the result is a tie, but either candidate could move slightly toward the median and win for sure. The same logic captures non-symmetric equilibria once we note that, because the electoral game is constant-sum, equilibria are interchangeable. Implicit in this logic is the assumption that candidates are perfectly informed about the location of the median voter. The political science literature on "probabilistic voting" relaxes the assumption of perfect information, assuming instead that candidates share a common prior distribution about the location of the median voter. It is a "folk theorem" that, in this environment, the unique Nash equilibrium is for both candidates to locate at the *median of the prior distribution* of the median voter's ideal point.[1] The median voter theorem therefore extends to the probabilistic voting model in an intuitive way. It is still implicitly assumed, however, that the candidates are symmetrically informed about the location of the median.

Symmetric information, while a useful simplifying assumption, is clearly very strong. We would expect private information about voter preferences to arise from differences in the candidates' personal experiences or from the different backgrounds of their political advisors. A better-documented source of private information is the use of private pollsters: 46 percent of all spending on U.S. Congressional campaigns in 1990 and 1992 was devoted to the hiring of political consultants, and the use of political pollsters was more common than any other type of consultant. Of 805 U.S. House of Representatives races in 1990, 209 campaigns employed private pollsters; of 856 races in 1992, 396 campaigns did. In races for open seats, the fraction of candidates using pollsters was about one half in both election cycles (Medvic (2001)). This is in addition to the polling services offered by the major parties, particularly the Republican party.

In this paper, we develop a general model of elections that allows for privately-informed candidates. Before selecting a platform, each candidate receives a signal drawn from an arbitrary finite set of possible signals; each candidate updates about the location of the median

[1]See Calvert (1985) for a result in this vein.

voter and the platform of the opponent and then chooses a platform; the median voter's location is then realized, and the candidate closest to the median wins. In the natural setting where candidates have access to identical polling technologies, we prove uniqueness and fully characterize the pure strategy equilibrium of the model, if it exists. We find that the logic of the median voter theorem *does not* extend to the general case in the expected way: After receiving a signal, a candidate updates the prior distribution of the median voter, *conditioning on both candidates receiving that same signal,* and locates at the median of *that* posterior distribution.[2] In the Downsian and probabilistic voting models, the candidates have symmetric information, and conditioning on one candidate receiving a signal is the same as conditioning on both receiving it, so we obtain the known results for those models as special cases. When information is asymmetric, though one might expect a candidate simply to target the median voter conditional on his/her own signal, this is not what happens. In fact, a candidate's equilibrium platform is a biased estimator of the median voter, and strategic competition between the candidates often leads the candidates to take positions more extreme than their estimates of the median voter's ideal point. That is, in stark contrast to the complete platform convergence in the Downsian and standard probabilistic voting models, private signals gives rise to strategic platform divergence.[3]

To understand this result, consider a symmetric pure strategy equilibrium. As we vary a candidate's platform after receiving some signal s, the candidate's probability of winning varies continuously, *except* when the other candidate chooses the same platform after s: In that case, continuity holds if and only if that platform is the median of the distribution conditional on both receiving s. But candidates could exploit any discontinuity by moving slightly toward the median conditional on two s signals, so the only possible symmetric pure strategy equilibrium is the one claimed. Finally, because the electoral game is constant-sum, symmetry and interchangeability imply that there are no other asymmetric pure strategy equilibria.

[2]This result is reminiscent of the findings of Milgrom (1981), who shows that in a common-value second-price auction, the equilibrium bid of a type θ corresponds to the expected value of the good conditional on both types being equal to θ. Here, since candicates maximize the probability of winning, the relevant statistic is the median.

[3]Policy convergence is a robust result in the Downsian model surviving even if candidates place weight on policy platforms (Calvert (1985), Duggan and Fey (2001)). Adding policy motivation to the probabilistic voting model leads to partial policy convergence, as in any pure strategy equilibrium, a wedge appears between the candidates' platforms (Wittman (1983), Calvert (1985)). Banks and Duggan (1999) show that convergence obtains quite generally in a related class of probabilistic voting models, where candidates seek to maximize plurality. Duggan (2000), Bernhardt, Hughson, and Dubey (2002), and Banks and Duggan (2002) find partial convergence in models of repeated elections. Finally, if one candidate has a "valence advantage," so that all voters would vote for that candidate even if the opponent offers a slightly better policy, then equilibria in pure strategies do not exist, but mixed strategy equilibria obviously lead candidates to adopt distinct platforms (Aragones and Palfrey (2002), Groseclose (2001)).

We give sufficient conditions for existence of the pure strategy equilibrium under two sets of conditions. First, we consider the case where candidates' information is coarse, in the sense that polling generates either a signal that the median voter is likely to the left or likely to the right. With only two possible signals, the key condition guaranteeing existence is quite weak: Conditional on a candidate receiving a signal, the probability that the opponent receives the same signal should be at least one half, i.e., signals should not be negatively correlated. We then allow for multiple signals, and the key condition becomes: Conditional on a candidate receiving a signal, the probability that the opponent receives a signal weakly to the "left" should exceed the probability that the opponent receives a signal strictly to the "right," and vice versa. This limits the incentive for a candidate to move away from the equilibrium platform after any signal, and, together with other background conditions, it ensures the existence of the pure strategy equilibrium. This key condition is clearly more difficult to sustain when the number of possible signals is large, and we show that the pure strategy equilibrium may fail to exist as a consequence. In fact, we construct an example of non-existence by adding an arbitrarily small amount of asymmetric information to the Downsian model in an appropriate way, raising the issue of robustness of the median voter theorem with respect to private information.

This leads us to analyze mixed strategy equilibria: We prove that mixed strategy equilibria exist; we prove that the (unique) mixed strategy equilibrium payoffs of our model vary continuously in its parameters; and we use this result to prove upper hemicontinuity of equilibrium mixed strategies. In the context of the probabilistic voting model, Ball (1999) proves the existence of a mixed strategy equilibrium, and our existence result can be viewed as extending his to models with asymmetrically-informed candidates. We then show that in models "close" to the Downsian model, mixed strategy equilibria must be "close" (in the sense of weak convergence) to the median voter's ideal point. Thus, in the Downsian model, while the pure strategy equilibrium may cease to exist when small amounts of asymmetric information are added, we regain robustness in mixed strategies: Mixed strategy equilibria exist, and they must all be close to the median.[4] Finally, we bound the support of mixed strategy equilibria by the interval defined by the smallest and largest conditional medians, and we show that the only possible atoms of equilibrium mixed strategies are at conditional medians.

Bernhardt, Duggan, and Squintani (2003) explicitly solve for a mixed strategy equilibrium

[4] Banks and Duggan (1999) prove a similar result in the probabilistic voting model with expected plurality-maximizing candidates. There, the unique pure strategy equilibrium may cease to exist when a small amount of randomness is introduced into voter behavior, but mixed strategy equilibria exist and vary continuously in this respect.

in a tractable version of our model, they conduct comparative statics analysis, and they investigate the effect of electoral competition on voter welfare. Ledyard (1989) first raised the issue of privately informed candidates and considered several examples exploring the effects of the order of candidate position-taking, public polls, and repeated elections. A number of other papers have independently considered aspects of elections with privately-informed candidates. Chan (2001) develops a related three-signal model, and assuming a pure strategy equilibrium exists, he provides partial characterizations and welfare analysis. Ottaviani and Sorensen (2003) consider a model of financial analysts who receive private signals of a firm's earnings and simultaneously announce forecasts, with rewards depending on the accuracy of their predictions. The case of two analysts can be interpreted as a model of electoral competition with privately informed candidates. They assume a normally distributed median, and offer an analysis of pure strategy equilibrium based on necessary first order conditions, but they do not address existence analytically. More distantly related are Heidhues and Lagerlöf (2001) and Martinelli (2001,2002).

2 The Electoral Framework

Two political candidates, A and B, simultaneously choose policy platforms, x and y, on the real line, $\Re$. We model the electorate as a cut point, μ, that determines which candidate wins: Candidate A wins the election if $|x - \mu| < |y - \mu|$ and loses if the inequality is reversed; if $|x - \mu| = |y - \mu|$, the election is decided by a fair coin toss, so that A wins with probability one half. Assuming symmetric voter utilities, this formulation captures representative voter models and, as long as a median is uniquely defined, models with multiple voters. In particular, the median is unique if there is an odd number of voters, or if there is a continuum of voters with ideal points distributed according to a density with convex support. In such models, policy z is majority-preferred to w if and only if z is preferred by the median voter, and we can identify the cut point μ with the median ideal point.

Candidates do not observe the location of μ, but receive private signals, s and t, about its true location. These signals may reflect the personal experiences of the candidates, or they may be generated by private polls conducted by the candidates' campaign organizations or political parties. Let S denote the finite set of possible signals for A, and let T denote the finite set of possible signals for B. Candidates have a common prior distribution on $\Re \times S \times T$, where the distribution of μ conditional on signals s and t is $F_{s,t}$, and the marginal probability of signal pair (s,t) is $P(s,t)$. The marginal probability of signal s is $P(s)$, and that of signal t is $P(t)$, where we assume that $P(s) > 0$ and $P(t) > 0$. Conditional probabilities $P(\cdot|s)$ and $P(\cdot|t)$ are then defined using Bayes rule. The model is completely general with respect to the correlation between candidates' signals, allowing for conditionally-independent signals and perfectly-correlated signals as special cases. We assume that each $F_{s,t}$ is continuous, and that for all $a,b,c \in \Re$ with $a < b < c$, $0 < F_{s,t}(a)$ and $F_{s,t}(c) < 1$ implies $F_{s,t}(a) < F_{s,t}(b) < F_{s,t}(c)$. Thus, $F_{s,t}$ admits a density, denoted $f_{s,t}$, with convex support. Let $m_{s,t}$ be the uniquely-defined median of $F_{s,t}$. Given a subset $T' \subseteq T$, we define $F_{s,T'}$ by

$$F_{s,T'}(z) = \sum_{t \in T'} \frac{P(t|s)}{P(T'|s)} F_{s,t}(z),$$

when $P(T'|s) > 0$, and we let $m_{s,T'}$ denote the unique median of $F_{s,T'}$. We define $F_{S',t}$ and $m_{S',t}$ analogously. We write F_s for $F_{s,T}$, the distribution of μ conditional only on s, and m_s for the associated median; F_t and m_t are defined analogously.

Thus, the probability that candidate A wins when A uses platform x and receives signal s and B uses platform y and receives signal t, is

$$F_{s,t}\Big(\frac{x+y}{2}\Big) \text{ if } x < y, \quad 1 - F_{s,t}\Big(\frac{x+y}{2}\Big) \text{ if } y < x, \quad \text{and} \quad \frac{1}{2} \text{ if } x = y.$$

The probability that B wins has an analogous form and, of course, is equal one minus the probability that A wins. This defines a Bayesian game between the candidates, in which pure strategies for the candidates are vectors $X = (x_s)$ and $Y = (y_t)$, and given pure strategies X and Y, candidate A's interim expected payoff conditional on signal s is

$$\begin{aligned}\Pi_A(X,Y|s) \;&=\; \sum_{t \in T: x_s < y_t} P(t|s) F_{s,t}\Big(\frac{x_s+y_t}{2}\Big) + \sum_{t \in T: y_t < x_s} P(t|s)\Big(1 - F_{s,t}\Big(\frac{x_s+y_t}{2}\Big)\Big) \\ &\quad + \frac{1}{2} \sum_{t \in T: x_s = y_t} P(t|s).\end{aligned}$$

Candidate B's interim expected payoff is defined analogously. A *pure strategy Bayesian equilibrium* is a strategy pair (X,Y) such that

$$\Pi_A(X,Y|s) \;\geq\; \Pi_A(X',Y|s),$$

for all signals $s \in S$ and all strategies X'; and

$$\Pi_B(X,Y|t) \;\geq\; \Pi_B(X,Y'|t),$$

for all signals $t \in T$ and all strategies Y'. This formalizes the idea that the candidates' campaign platforms are optimal given all information available to them.

We define candidate A's *ex ante* expected payoff, the expected payoff before receiving a signal, as

$$\Pi_A(X,Y) \;=\; \sum_{s\in S} P(s)\Pi_A(X,Y|s),$$

and candidate B's *ex ante* expected payoff, $\Pi_B(X,Y)$, is defined analogously. Clearly,

$$\Pi_A(X,Y)+\Pi_B(X,Y) \;=\; 1,$$

for all strategies X and Y. Note that (X,Y) is a pure strategy Bayesian equilibrium if and only if

$$\Pi_A(X,Y') \;\geq\; \Pi_A(X,Y) \;\geq\; \Pi_A(X',Y),$$

for all X' and all Y'. Thus, the pure strategy Bayesian equilibria are equilibria of a two-player, constant-sum game. The constant sum property implies that equilibria are "interchangeable": If (X,Y) and (X',Y') are equilibria, then so are (X,Y') and (X',Y).

At times, we impose additional conditions. Conditions (C1)-(C4) define our *Canonical Model* of polling, in which candidates employ identical polling technologies and signals exhibit a natural ordering structure. We first impose symmetry conditions (C1) and (C2).

(C1) $S=T$.

Thus, the same set I, with elements i, j, etc., can be used to index these sets. We then write $P(i,j)$ for $P(s_i,t_j)$, $F_{i,j}$ for F_{s_i,t_j}, and so on.

(C2) For all signals $i,j\in I$, $P(i,j)=P(j,i)$ and $F_{i,j}=F_{j,i}$.

Condition (C2) implies that signal i of candidate A can be identified with signal i of candidate B in the sense that they are equally informative. While the general model allows for asymmetries between candidates, it is natural to expect candidates to have equal access to polling technologies, in which case conditions (C1) and (C2) are appropriate. In that case, we will be especially interested in equilibria in which candidates use information similarly: A *symmetric* pure strategy Bayesian equilibrium is an equilibrium (X,Y) in which $x_i=y_i$ for all $i\in I$.

Under (C1)-(C2), candidates' *ex ante* payoffs are symmetric in the sense that

$$\Pi_A(X,Y) \;=\; \Pi_B(Y,X),$$

for all X and Y. Thus, pure strategy Bayesian equilibria of electoral competition are equilibria of a two-player, symmetric, constant sum game. Symmetry implies that if (X, Y) is an equilibrium, then so is (Y, X). With interchangeability, we conclude that if (X, Y) is an equilibrium, then (X, X) and (Y, Y) are symmetric equilibria.

The next condition, which assumes (C1), says that if one candidate receives a signal, then it must also be possible for the other candidate to receive it.

(C3) For all signals $i \in I$, $P(i, i) > 0$.

The last condition defining the Canonical Model imposes a natural ordering structure on signals. Again assume (C1) holds.[5]

(C4) There exists an ordering $\precsim$ of I, with asymmetric part $\prec$ and symmetric part $\sim$, such that for all signals $i, j \in I$:

(i) $i \prec j$ if and only if, for all $K \subseteq I$ with $P(K|i)P(K|j) > 0$, we have $m_{i,K} < m_{j,K}$.

(ii) $i \sim j$ if and only if, for all $k \in I$, $F_{i,k} = F_{j,k}$.

Several implications of (C4) for the Canonical Model are noteworthy. If $m_{i,K} = m_{j,K}$ for one subset K with $P(K|i) > 0$ and $P(K|j) > 0$, then $i \sim j$. In that case, $F_{i,i} = F_{j,i} = F_{i,j} = F_{j,j}$, so that $i \sim j$ implies $m_{i,i} = m_{j,j}$. If $i \prec j$, then the conditional medians satisfy

$$m_{i,\{i\}} < m_{j,\{i\}} = m_{i,\{j\}} < m_{j,\{j\}},$$

so that $i \prec j$ implies $m_{i,i} < m_{j,j}$. Combining these observations reveals that $i \prec j$ if and only if $m_{i,i} < m_{j,j}$.

Condition (C4) is natural if "higher" signals are correlated with higher values of the cut point. The second part of the condition says that two signals are equivalently ranked by $\precsim$ only if they generate the same conditional distributions. It is trivially satisfied if $\precsim$ is "linear," i.e., if $i \neq j$ implies $i \prec j$ or $j \prec i$, which is a reasonable assumption in many applications. By admitting the possibility of conditional equivalence, however, we allow for candidates to condition their platforms on informationally-redundant signals: The same consultant's report

[5] In the following definition, we call $\precsim$ an ordering to indicate that it is complete and transitive. The asymmetric part $\prec$ is defined as $i \prec j$ if and only if $i \precsim j$ and not $j \precsim i$, and the symmetric part $\sim$ is defined as $i \sim j$ if and only if $i \precsim j$ and $j \precsim i$. We write $i \succsim j$ if $j \precsim i$, and similarly for $\succ$.

received in the morning or in the afternoon will convey the same information, but a candidate could conceivably condition his/her platform on such informationally-irrelevant detail; collapsing the two reports into one signal implicitly rules out this kind of conditioning.

The *Downsian Model* of elections is characterized by two features, one that specializes our model and another that places it outside the class of models we consider here. The first feature is that the candidates have *complete information* about each other's signals: if $P(s,t) > 0$, then $P(t|s) = P(s|t) = 1$. In particular, the conditional distribution of the cut point, $F_{s,t}$, is common knowledge between the candidates, and the conditional median $m_{s,t}$ following signal s is m_s with probability one. The second feature is that, conditional on a candidate's signal (or equivalently, conditional on a pair (s,t)), the location of the cut point is known with certainty. Formally, $F_{s,t}$ is the point mass on $m_{s,t}$, violating our maintained assumption that $F_{s,t}$ is continuous. A candidate then wins with probability one if his platform is closer to the cut point than the other candidate's, generating expected payoffs for A of:

$$\Pi_A(x_s, y_t|s) \quad = \quad \begin{cases} 1 & \text{if } x_s < y_t \text{ and } \frac{x_s+y_t}{2} < m_s \\ 1 & \text{if } y_t < x_s \text{ and } m_s < \frac{x_s+y_t}{2} \\ \frac{1}{2} & \text{if } x_s = y_t \text{ or } \frac{x_s+y_t}{2} = m_s \\ 0 & \text{else,} \end{cases}$$

where $P(t|s) = 1$. By the median voter theorem, the unique pure strategy equilibrium is that candidate A locates at m_s after receiving signal s and candidate B locates at m_t after receiving signal t. A consequence of Banks, Duggan, and Le Breton's (2001) Theorem 4 is that this equilibrium is indeed unique among all mixed strategy equilibria as well.

Many of our results extend without modification to the Downsian Model, but significant difficulties arise if complete information is weakened or if further discontinuities in conditional distributions, even of a very limited sort, are allowed. We will illustrate some of these difficulties in an extension of the model, the *Generalized Downsian Model*, in which we drop complete information and assume only that for each $s, t \in T$ with $P(s,t) > 0$, $F_{s,t}$ is the point mass on $m_{s,t}$. Of course, we can approximate a point mass with continuous distributions, so we obtain the Downsian Model as a "limit point" of the class of models we consider, allowing us to take up the robustness of the median voter theorem when the Downsian Model is perturbed by adding small amounts of private information.

3 Special Cases of the Electoral Model

We now present four special cases of the electoral model.

Common Knowledge Model. Here, we consider the situation in which following every signal pair (s, t) that occurs with positive probability, each candidate knows the conditional distribution of the cut point with certainty, and each knows that the other knows the conditional distribution with certainty, and so on. That is, the conditional distribution is common knowledge. Formally, for all $s \in S$ and all $t \in T$, $P(s, t) > 0$ implies that:

$$F_{s,t'} \quad = \quad F_{s',t},$$

for all $t' \in T$ with $P(t'|s) > 0$ and all $s' \in S$ with $P(s'|t) > 0$. Thus, $F_s = F_{s,t}$ for all $t \in T$, and similarly $F_t = F_{s,t}$ for all $s \in S$. Write $s \sim_S \hat{s}$ if $F_s = F_{\hat{s}}$, and note that $\sim_S$ is an equivalence relation. We define the equivalence relation $\sim_T$ on T analogously. We partition S into equivalence classes

$$S(s) \quad = \quad \{\hat{s} \in S \mid s \sim_S \hat{s}\},$$

and we write $F_{S(s)} = F_s$ for the conditional distribution determined by the signals in $S(s)$. Similarly, we partition T into equivalence classes

$$T(t) \quad = \quad \{\hat{t} \in T \mid t \sim_T \hat{t}\},$$

writing $F_{T(t)} = F_t$, and we use $\hat{S}$ and $\hat{T}$ to denote generic equivalence classes. It is clear that $P(\hat{S} \times \hat{T}) > 0$ if and only if $F_{\hat{S}} = F_{\hat{T}}$.

The electoral game with sets $\hat{S}$ and $\hat{T}$ of signals can be analyzed independently: The strategy pair (X, Y) is a pure strategy Bayesian equilibrium if and only if for each pair $\hat{S}$ and $\hat{T}$ with $P(\hat{S} \times \hat{T}) > 0$, the restricted strategies $(X_{\hat{S}}, Y_{\hat{T}}) = (x_s, y_t)_{s \in \hat{S}, t \in \hat{T}}$ form an equilibrium of the restricted game with payoffs

$$\begin{aligned} \Pi_A(X_{\hat{S}}, Y_{\hat{T}}|s) \quad = \quad & \sum_{t \in \hat{T}: x_s < y_t} P(t|s) F\left(\frac{x_s + y_t}{2}\right) + \sum_{t \in \hat{T}: y_t < x_s} P(t|s)\left(1 - F\left(\frac{x_s + y_t}{2}\right)\right) \\ & + \frac{1}{2} \sum_{t \in \hat{T}: x_s = y_t} P(t|s), \end{aligned}$$

for all $s \in \hat{S}$, where $F = F_{\hat{S}} = F_{\hat{T}}$, and likewise for candidate B. We call this restricted game the *component game* of $\hat{S} \times \hat{T}$.

If (C1)-(C3) are imposed in the Common Knowledge Model, (C4) holds if and only if distinct equivalence classes $\hat{I}$ and $\tilde{I}$ are associated with distinct medians, i.e., $\hat{I} \neq \tilde{I}$ implies $m_{\hat{I}} \neq m_{\tilde{I}}$. In this case, the ordering $\precsim$ on I is such that lower signal realizations correspond to equivalence classes with lower medians, so that the relation $\sim_I$ is the symmetric part of $\precsim$.

A familiar logic suggests that locating at the median m_s after receiving signal s is a compelling strategy, leading to the following simple characterization of pure strategy Bayesian equilibria under common knowledge. Note that, in equilibrium, each candidate takes the same position as the other with probability one.

Proposition 1 *In the Common Knowledge Model, the strategy pair* (X, Y) *is a pure strategy Bayesian equilibrium if and only if, for all* $s \in S$ *and all* $t \in T$, $x_s = m_s$ *and* $y_t = m_t$. *Thus,* $P(s,t) > 0$ *implies* $x_s = y_t = m_{s,t}$.

Proof: Let (X, Y) be a pure strategy Bayesian equilibrium. Take any $s \in S$, consider a pure strategy $\hat{X}$ such that $\hat{x}_s = m_s$, and note that, for every strategy Y', $\Pi_A(\hat{X}, Y|s) \geq \frac{1}{2}$. Thus, candidate A's expected payoff conditional on any signal s cannot be less than one half in equilibrium. Similarly defining $\hat{Y}$ such that $\hat{y}_t = m_t$, candidate B's expected payoff conditional on any signal t is at least one half. Now suppose $P(s,t) > 0$, so that $m_s = m_t$, and suppose that $x_s > m_s = m_t$. Then, however, $\Pi_B(X, \hat{Y}) > \frac{1}{2}$, since $F_t\left(\frac{m_t + x_s}{2}\right) > \frac{1}{2}$. Since Y is a best response to X, we must also have $\Pi_B(X, Y) > \frac{1}{2}$, which implies $\Pi_A(X, Y) < \frac{1}{2}$, a contradiction. A similar argument holds for $x_s < m_s$. Now define X by $x_s = m_s$ for all $s \in S$ and Y by $y_t = m_t$ for all $t \in T$. Since $P(t|s) > 0$ implies $m_t = m_s$, we see that $\Pi_A(X, Y|s) = \frac{1}{2}$ and, for all other strategies X', that $\Pi_A(X', Y|s) \leq \frac{1}{2}$. An analogous argument for candidate B establishes that (X, Y) is an equilibrium. The last claim in the proposition then follows immediately from common knowledge. ∎

While the necessity part of the proposition is stated only for pure strategy equilibria, a consequence of later results is that the characterization for equilibria under common knowledge extends to mixed strategy equilibria as well.

Probabilistic Voting Model. We capture the traditional probabilistic voting model with an unknown median voter as the special case of the Canonical Model in which information between the candidates is complete, meaning that $P(i|i) = 1$ for all $i \in I$. In fact, this is also a special case of the Common Knowledge Model, and from the observations above, it follows that the electoral game following signal pair (i, i) can be analyzed independently as the game with strategy set $\Re$ for each candidate and payoffs

$$\Pi_A(x_i, y_i|i) = \begin{cases} F_{i,i}\left(\frac{x_i + y_j}{2}\right) & \text{if } x_i < y_i \\ 1 - F_{i,i}\left(\frac{x_i + y_j}{2}\right) & \text{if } y_i < x_i \\ \frac{1}{2} & \text{if } x_i = y_i. \end{cases}$$

The unique pure strategy Bayesian equilibrium is given by Proposition 1, with the candidates locating at m_i after signal i, confirming the "probabilistic" version of the median voter theorem from the theory of probabilistic voting. The model is usually defined without reference to a realized signal pair, essentially taking that as given and suppressing the informational foundations of the model.

Stacked-Uniform Model. This special case of the Canonical Model captures situations with a number of "preliminary" locations of the median voter, where candidate polling generates signals about these preliminary locations, and subsequent to polling there is a uniform disturbance to the location of the median voter. Let $\mu = \alpha + \beta$, where α is uniformly distributed on $[-a, a]$ and β is an independently-distributed discrete random variable with support on $b_1 < b_2 < \cdots < b_N$. Let $Q(b_k)$ denote the probability of b_k.Candidates share the same set of signals. Signals depend stochastically on the realization of β:

$$P(i,j) = \sum_{k=1}^{N} Q(i,j|b_k)Q(b_k),$$

where $Q(i, j|b_k)$ is the probability conditional on b_k that candidates receive signals i and j. Given b_k, μ is uniformly distributed on $[b_k - a, b_k + a]$, with piece-wise linear distribution

$$F_k(z) = \max\left\{\min\left\{0, \frac{z - b_k + a}{2a}\right\}, 1\right\}.$$

By Bayes rule, the probability of b_k conditional on signals i and j is

$$Q(b_k|i,j) = \frac{Q(i,j|b_k)Q(b_k)}{P(i,j)}$$

when well-defined, and the distribution of μ conditional on signals i and j is

$$F_{i,j}(z) = \sum_{k=1}^{N} Q(b_k|i,j)F_k(z).$$

Condition (C2) is satisfied if $Q(i, j|b_k) = Q(j, i|b_k)$ for all i, j, and k. Condition (C3) holds if, for all $i \in I$, there exists a b_k such that $Q(i, i|b_k)Q(b_k) > 0$.

For general numbers of preliminary cut-point locations and signal realizations, our equilibrium characterizations are sharpest when a is large, specifically $a \geq b_N - b_1$. Then $b_N - a \leq b_1 + a$, so that conditional densities are single-plateaued. Indeed, $a \geq b_N - b_1$ implies that the conditional medians lie in the center interval: For all $i \in I$, $b_N - a \leq m_{i,i} \leq b_1 + a$. To see this, note that $Q(b_1|i, i) = 1$ yields a lower bound for $m_{i,i}$. In that case, $F_{i,i}$ is uniformly distributed with median $b_1 \geq b_N - a$. Therefore, $b_N - a \leq m_{i,i}$. Similarly, $Q(b_n|i, i) = 1$ provides an upper bound for $m_{i,i}$. In that case, $F_{i,i}$ is uniformly distributed with median $b_N \le b_1 + a$. Thus, $m_{i,i} \le b_1 + a$, as claimed. As a consequence,

$$F_{i,j}(z) \;=\; \sum_{k=1}^{N} Q(b_k|i,j)\left(\frac{z+a-b_k}{2a}\right) \;=\; \frac{1}{2} + \frac{z - \sum_{k=1}^{N} Q(b_k|i,j)b_k}{2a},$$

for all $z \in [b_N - a, b_1 + a]$. Therefore, when $a \ge b_N - b_1$,

$$m_{i,j} \;=\; \sum_{k=1}^{N} Q(b_k|i,j)b_k.$$

Substituting for $m_{i,j}$ into $F_{i,j}$ using equation (1) yields

$$F_{i,j}(z) \;=\; \frac{a - m_{i,j} + z}{2a},$$

for all $z \in [b_N - a, b_1 + a]$, and hence,

$$F_{i,j}(z) - F_{i,j}(w) \;=\; \frac{z - w}{2a},$$

for all $z, w \in [b_N - a, b_1 + a]$. Note that, from (C4), $m_{h,i} < m_{j,k}$ implies that $F_{h,i}(z) > F_{j,k}(z)$ for all $z \in [b_N - a, b_1 + a]$, i.e., $F_{j,k}$ stochastically dominates $F_{h,i}$ over the interval.

In the next section, we give a complete characterization of the unique pure strategy Bayesian equilibrium, if it exists. See Bernhardt, Duggan, and Squintani (2003) for an explicit solution of the (essentially unique) mixed strategy equilibrium of the Stacked-Uniform Model when the pure strategy equilibrium fails to exist.

Shape-Invariant Model. In this special case of the Canonical Model, the shape of the distribution of μ conditional on pairs of signal realizations is the same regardless of the signal realizations. This is equivalent to identifying the median conditional on the signal realizations as a scale parameter of a family of shape-invariant distributions. Formally, we assume that

$$F_{i,j}(z + m_{i,j}) \;=\; F_{k,\ell}(z + m_{k,\ell}),$$

for all $z \in \Re$ and all $i, j, k, \ell \in I$. The assumption that conditional distributions form a family of shape-invariant distributions is pervasive in statistics—the most celebrated example of such a family is the homoskedastic econometric models with normally-distributed errors.

4 Pure Strategy Equilibrium

Proposition 1 characterized the unique pure strategy Bayesian equilibrium of the model when the conditional distribution of the cut point was common knowledge. The main result of this section is a full characterization of the pure strategy equilibria of the Canonical Model: If a pure strategy Bayesian equilibrium exists, then it is unique; and after receiving a signal, a candidate locates at the median of the distribution of μ conditional on *both* candidates receiving that signal. This is consistent with our result for common knowledge, because under that assumption the distribution of the cut point conditional on one i signal is the same as that conditional on two i signals, i.e., $m_i = m_{i,i}$. However, Proposition 1 extends to the Canonical Model in a perhaps unexpected way. Given a natural restriction on conditional medians, a corollary is that candidates take policy positions that are extreme relative to their expectations of μ given their own information. In other words, a candidate's location is a biased estimator of μ: Candidates who receive high signals overshoot μ, while those who receive low signals undershoot. We also provide sufficient conditions for the existence of a pure strategy equilibrium.

The characterization result relies on Lemma 3, which we prove in the appendix.

Lemma 3 *In the Canonical Model, let (X, Y) be a pure strategy Bayesian equilibrium. If $x_i = y_j$ for some $i, j \in I$ with $P(i, j) > 0$, then $x_i = y_j = m_{i,j}$.*

The intuition behind this lemma is simple. Suppose that candidates A and B locate at the same point following two signal realizations, i and j, and suppose for simplicity that these are the only realizations for which they locate there; the proof uses (C4) to rule out other cases. Then, conditional on those realizations, each candidate expects to win the election with probability one half. If the candidates are not located at the median conditional on signals i and j, then the payoff of either candidate would be increased by a small move toward that conditional median: If A deviates in this way, then A's expected payoff given other signal realizations for B varies continuously with A's location, but A's payoff given realization j would jump discontinuously above one half. Therefore, a slight deviation would raise A's payoff, something that is impossible in equilibrium.

Theorem 1 (Necessity) *In the Canonical Model, if (X, Y) is a pure strategy Bayesian equilibrium, then $x_i = y_i = m_{i,i}$ for all $i \in I$.*

Proof: First, consider a symmetric equilibrium (X, Y), where $x_i = y_i$ for all $i \in I$. By (C3) and Lemma 3, $x_i = y_i = m_{i,i}$, as required. Now suppose there is an asymmetric equilibrium (X, Y), where $x_i \neq m_{i,i}$ for some $i \in I$, and define the strategy $Y' = X$ for candidate B.

Then, by symmetry and interchangeability, (X, Y') is a symmetric Bayesian equilibrium with $x_i \neq m_{i,i}$, contradicting Lemma 3. ∎

In many situations, it is reasonable to suppose that lower signals indicate lower values of μ and higher signals indicate higher ones. Then Theorem 1 implies that polling leads candidates to extremize their locations.

Corollary 1 *In the Canonical Model, suppose there exists a signal $c \in I$ such that $i \prec c$ implies $m_{i,i} < m_i$ and $c \prec i$ implies $m_i < m_{i,i}$. If (X, Y) is a pure strategy Bayesian equilibrium, then $x_i < m_i$ for $i \prec c$ and $m_i < x_i$ for $i \succ c$.*

This result is sufficiently important to highlight. If c corresponds to an uninformative signal in the sense that the conditional median $m_{c,c}$ is equal to the unconditional median, then it follows that in a pure strategy equilibrium, private polling magnifies platform divergence: Candidates bias their locations in the direction of their private signals past the median given their own signal and away from the unconditional median.

Theorem 1 gives a necessary, not a sufficient, condition for the existence of a pure strategy equilibrium. While we soon give sufficient conditions for existence, the next example shows that pure strategy equilibria do not always exist. In fact, the example demonstrates Canonical Models arbitrarily close to the Downsian Model in which choosing $m_{i,i}$ after signal i is not an equilibrium. An implication of Theorem 1 is that pure strategy equilibria may not exist in Canonical Models arbitrarily close to the Downsian Model, demonstrating a type of fragility of the Downsian equilibrium. We return to this issue in our analysis of mixed strategies. An implication of results there is that even if pure strategy equilibria fail to exist in models close to the Downsian Model, mixed strategy equilibria do exist and will necessarily be "close" to the Downsian equilibrium.

Example 1 *Fragility of Pure Strategy Equilibrium in the Downsian Model.* Consider the Downsian Model in which $I = \{-1, 1\}$ and $P(-1,-1) = P(1,1) = \frac{1}{2}$, with conditional distributions $F_{-1,-1}$ and $F_{1,1}$ with point mass on $m_{-1,-1}$ and $m_{1,1} = m_{-1,-1} + 1$, respectively. Let $F_{-1,1}$ and $F_{1,-1}$ be degenerate on $m_{-1,1} = m_{1,-1} = \frac{m_{-1,-1}+m_{1,1}}{2}$. By the median voter theorem, the unique equilibrium is (X, Y) defined by $x_{-1} = y_{-1} = m_{-1,-1}$ and $x_1 = y_1 = m_{1,1}$. Now define the sequences $\{F^n_{i,j} \mid i, j = -1, 1\}$ of conditional distributions as follows. For each $n \geq 2$, let $P^n(-1,-1) = P^n(1,1) = \frac{1}{2} - \frac{1}{n}$ and $P^n(-1,1) = P^n(1,-1) = \frac{1}{n}$. Let $F^n_{-1,-1}$ be the uniform distribution on $\Re$ with density n centered at $m_{-1,-1}$; let $F^n_{1,1}$ be the uniform distribution with density n centered at $m_{1,1}$; and let $F^n_{-1,1} = F^n_{1,-1}$ be the uniform distribution with density n^2 centered at $m_{-1,1} = m_{1,-1}$. Note that the conditional medians of $F^n_{-1,-1}$ and $F^n_{1,1}$ are fixed at $m_{-1,-1}$ and $m_{1,1}$, respectively, for all n. Furthermore, the upper bound of the support of $F^n_{-1,1} = F^n_{1,-1}$ is $m_{-1,1} + \frac{1}{2n^2}$. By Theorem 1, the only possible pure strategy Bayesian equilibrium in the n^{th} game is (X, Y) defined above. But we claim that (X, Y) is not an equilibrium, because A can deviate profitably to strategy $\hat{X}^n$ defined by $\hat{x}^n_{-1} = m_{-1,-1} + \frac{1}{n^2}$ and $\hat{x}^n_1 = x^n_1$. To see this, note that

$$\begin{aligned} &\Pi_A(\hat{X}^n, Y|-1) - \Pi_A(X, Y|-1) \\ &\quad = P^n(-1|-1)\left[1 - F^n_{-1,-1}\left(\frac{m_{-1,-1} + \hat{x}^n_{-1}}{2}\right)\right] + P^n(1|-1)(1) - \frac{1}{2}, \end{aligned}$$

where we use the fact that $m_{-1,1} + \frac{1}{2n^2} = \frac{1}{2}\left(m_{-1,-1} + \frac{1}{n^2} + m_{1,1}\right)$, which in turn implies $F^n_{-1,1}\left(\frac{\hat{x}^n_{-1} + m_{1,1}}{2}\right) = 1$. After substituting, this equals

$$\left(1 - \frac{2}{n}\right)\left(\frac{1}{2} - \frac{1}{2n}\right) + \frac{2}{n} - \frac{1}{2} = \frac{1}{2n} + \frac{1}{n^2} > 0,$$

establishing the claim. Since $F^k_{i,j} \to F_{i,j}$ weakly and $P^k(i,j) \to P(i,j)$ for $i, j = 1, -1$, the sequence of perturbed models can be chosen arbitrarily close to the Downsian Model. Thus, even the introduction of an arbitrarily small amount of private information into the Downsian Model can lead to the non-existence of pure strategy equilibrium. □

We next provide conditions that ensure existence of the pure strategy equilibrium characterized in Theorem 1. We consider two cases of the Canonical Model: with two possible signals our condition is weak, but in the multi-signal case we require more structure. In both cases, loosely speaking, a pure strategy equilibrium exists if conditional on receiving signal i, the probability that the other candidate also receives signal i is high enough.

Theorem 2 (Sufficiency for Binary Signals) *In the Canonical Model, let $I = \{-1, 1\}$. Then a sufficient condition for the unique pure strategy Bayesian equilibrium to exist is that*

$$\begin{aligned} P(1|1) f_{1,1}\left(\frac{z + m_{1,1}}{2}\right) &\geq P(-1|1) f_{1,-1}\left(\frac{z + m_{-1,-1}}{2}\right) \\ P(-1|-1) f_{-1,-1}\left(\frac{z + m_{-1,-1}}{2}\right) &\geq P(1|-1) f_{1,-1}\left(\frac{z + m_{1,1}}{2}\right), \end{aligned}$$

for all $z \in [m_{-1,-1}, m_{1,1}]$. In that equilibrium, candidates locate at $m_{i,i}$ following signal $i \in I$.

Proof: We show that (X,Y) is an equilibrium, where $x_i = y_i = m_{i,i}$ for $i = -1, 1$. Consider candidate A's best response problem, conditional on signal 1. If A deviates to $x \in [m_{-1,-1}, m_{1,1}]$, then the change in A's interim expected payoff is

$$\begin{aligned} & P(-1|1)\left[F_{1,-1}\left(\frac{m_{1,1}+m_{-1,-1}}{2}\right) - F_{1,-1}\left(\frac{x+m_{-1,-1}}{2}\right)\right] + P(1|1)\left[F_{1,1}\left(\frac{x+m_{1,1}}{2}\right) - \frac{1}{2}\right] \\ = & \int_x^{m_{1,1}} \left[P(-1|1)f_{1,-1}\left(\frac{z+m_{-1,-1}}{2}\right) - P(1|1)f_{1,1}\left(\frac{z+m_{1,1}}{2}\right)\right] dz \\ \le & \ 0. \end{aligned}$$

Thus, the deviation does not increase A's expected payoff. It is easily verified that deviations $x < m_{-1,-1}$ and $x > m_{1,1}$ are also unprofitable. A similar argument holds for signal $i = -1$, and a symmetric argument for candidate B establishes that (X,Y) is an equilibrium. ∎

The sufficient condition detailed in Theorem 2 is weak. If signals are not negatively correlated, so that $P(1|1) \ge P(-1|1)$ and $P(-1|-1) \ge P(1|-1)$, then a pure strategy equilibrium exists if

$$f_{1,1}\left(\frac{z+m_{1,1}}{2}\right) \ \ge \ f_{1,-1}\left(\frac{z+m_{-1,-1}}{2}\right) \tag{1}$$

for all $z \in [m_{-1,-1}, m_{1,1}]$, and

$$f_{-1,-1}\left(\frac{z+m_{-1,-1}}{2}\right) \ \ge \ f_{1,-1}\left(\frac{z+m_{1,1}}{2}\right) \tag{2}$$

for all $z \in [m_{-1,-1}, m_{1,1}]$. Inequality (1) compares $f_{1,1}$, shifted to the left by $\frac{m_{1,1}}{2}$, with $f_{1,-1}$ shifted to the left by $\frac{m_{-1,-1}}{2}$.

To make the nature of the comparison clearer, note that inequality (1) holds over $[m_{-1,-1}, m_{1,1}]$ if and only if

$$f_{1,1}\left(\frac{m_{1,1}-m_{-1,-1}}{2} + z\right) \ \ge \ f_{1,-1}(z) \tag{3}$$

holds over $\left[\frac{3m_{-1,-1}-m_{1,1}}{2}, \frac{m_{1,1}+m_{-1,-1}}{2}\right]$. Thus, the sufficient condition is that $f_{1,1}$ shifted to the left by $\frac{m_{1,1}-m_{-1,-1}}{2} > 0$ weakly exceed $f_{1,-1}$ over this range. This sufficient condition holds for the Stacked-Uniform Model with $a > b_N - b_1$, as conditional densities all equal $\frac{1}{2a}$ over the relevant range. The condition also holds in the Shape-Invariant Model when $f_{1,-1}$ is the translation of $f_{1,1}$ with median $\frac{m_{1,1}+m_{-1,-1}}{2}$. More generally, the condition holds if $f_{1,-1}$ has its median near $\frac{m_{1,1}+m_{-1,-1}}{2}$ and is somewhat more dispersed than $f_{1,1}$, as one would expect if identical signals decrease the variance of the distribution of μ and opposing signals, which offset each other, lead to a higher variance.

To simplify our sufficiency argument for existence in the multi-signal Canonical Model, we provide separate conditions on the priors over signal pairs and on the distribution of μ conditional on signal realizations. The statements of the conditions (C5)-(C7) are predicated on the structure of the Canonical Model. Condition (C5) is a regularity condition on the conditional distributions that reinforces the symmetry already present in the Canonical Model.

(C5) For all signals $i, j \in I$ with $P(i, j) > 0$, we have $m_{i,j} = \frac{m_{i,i}+m_{j,j}}{2}$.

Condition (C5) is trivially satisfied in the Common Knowledge Model. It is satisfied in the Stacked-Uniform Model with $a \geq b_N - b_1$ if

$$Q(b_k|i,j) \quad = \quad \frac{Q(b_k|i,i) + Q(b_k|j,j)}{2},$$

or equivalently if

$$\frac{Q(i,j|b_k)}{P(i,j)} \quad = \quad \frac{1}{2}\left(\frac{Q(i,i|b_k)}{P(i,i)} + \frac{Q(j,j|b_k)}{P(j,j)}\right),$$

for each b_k. We also impose a stochastic dominance-like restriction on the conditional distributions.

(C6) For all signals $i, j, k \in I$ with $i \precsim j \precsim k$, we have

$$F_{i,j}\left(\frac{m_{i,i}+z}{2}\right) \quad \geq \quad F_{j,k}\left(\frac{m_{k,k}+z}{2}\right) \tag{4}$$

for all $z \in [m_{i,i}, m_{k,k}]$.

Note that, in the Canonical Model, $i \precsim j \precsim k$ implies $m_{i,i} \leq m_{j,j} \leq m_{k,k}$, so the range over which inequality (4) holds is necessarily nonempty. To interpret the inequality, note that it holds over $[m_{i,i}, m_{k,k}]$ if and only if

$$F_{i,j}(z) \quad \geq \quad F_{j,k}\left(\frac{m_{k,k}-m_{i,i}}{2} + z\right)$$

holds over $\left[\frac{m_{i,i}}{2}, \frac{m_{i,i}+m_{k,k}}{2}\right]$. That is, the distribution conditional on signals j and k when shifted to the left by $\frac{m_{k,k}-m_{i,i}}{2} \geq 0$ must dominate the distribution conditional on signals i

and j. Condition (C6) is stronger than stochastic dominance in that $F_{j,k}$ is shifted to the left, but it is weaker in that the inequality must hold only over a given range. Again, the condition is trivially satisfied in the Common Knowledge Model, and at the end of this section we will show that it holds in the Stacked-Uniform Model when the disturbance term has a sufficiently large support, and in the Shape-Invariant Model when (C5) is satisfied.

Finally, we impose a restriction on priors over signals, formalizing the idea that conditional on a candidate's own signal, the probability the other candidate receives the same signal is sufficiently high. In fact, the condition is weaker than that, because it only restricts "net" probabilities.

(C7) For all signals $i \in I$,

$$\sum_{j \in I: j \precsim i} P(j|i) \ \geq \sum_{j \in I: j \succ i} P(j|i) \quad \text{and} \quad \sum_{j \in I: j \prec i} P(j|i) \ \leq \sum_{j \in I: j \succsim i} P(j|i).$$

An equivalent statement of (C7) is that $\sum_{j \in I: j \precsim i} P(j|i) \geq \frac{1}{2}$ and $\sum_{j \in I: i \precsim j} P(j|i) \geq \frac{1}{2}$. In words, for any signal i, it must be that i is a "median" of the distribution $P(\cdot|i)$ on I. Clearly, (C7) is most restrictive for the "extremal" signals, for which $P(i|i) \geq \frac{1}{2}$ is implied by the condition, and its restrictiveness depends on the number of possible signals. In the binary signal model, for example, it is satisfied whenever signals are not negatively correlated. Note also that (C7) is trivially satisfied in the Common Knowledge Model, because given any $i \in I$, the only signals j with $P(j|i) > 0$ are such that $i \sim j$.

Theorem 3 (Sufficiency for Multiple Signals) *In the Canonical Model, conditions (C5)-(C7) are sufficient for the existence of the unique pure strategy Bayesian equilibrium. In that equilibrium candidates locate at $m_{i,i}$ following signal $i \in I$.*

Proof: We show that (X, Y) is an equilibrium, where $x_i = y_i = m_{i,i}$ for all $i \in I$. Without loss of generality, we focus on candidate B's best response problem after receiving signal j. Consider a deviation to strategy Y'. There are two cases: $y'_j < m_{j,j}$ and $m_{j,j} < y'_j$. In the first case, define

$$\mathcal{G} = \{i \in I : m_{i,i} \leq y'_j\} \quad \text{and} \quad \mathcal{L} = \{k \in I : m_{j,j} \leq m_{k,k}\}.$$

Note that for all $i \in I \setminus (\mathcal{G} \cup \mathcal{L})$, we have $y'_j < m_{i,i} < m_{j,j}$. Hence, for i with $P(i|j) > 0$,

$$F_{i,j}\left(\frac{y'_j + m_{i,i}}{2}\right) - \left[1 - F_{i,j}\left(\frac{m_{i,i} + m_{j,j}}{2}\right)\right] \ \leq \ 0,$$

where we use (C5) to deduce that $F_{i,j}\left(\frac{y'_j+m_{i,i}}{2}\right) \leq \frac{1}{2}$ and $F_{i,j}\left(\frac{m_{i,i}+m_{j,j}}{2}\right) = \frac{1}{2}$. That is, B's gains from deviating when A receives signal $i \in I \setminus (\mathcal{G} \cup \mathcal{L})$ are non-positive. Therefore, the change in B's interim expected payoff satisfies

$$\begin{aligned}
\Pi_B(X, Y'|j) \quad - \quad \Pi_B(X,Y|j) &\leq \sum_{i\in\mathcal{G}} P(i|j)\left[1 - F_{i,j}\left(\frac{m_{i,i}+y'_j}{2}\right) - \left(1 - F_{i,j}\left(\frac{m_{i,i}+m_{j,j}}{2}\right)\right)\right] \\
&+ \sum_{k\in\mathcal{L}} P(k|j)\left[F_{j,k}\left(\frac{y'_j+m_{k,k}}{2}\right) - F_{j,k}\left(\frac{m_{j,j}+m_{k,k}}{2}\right)\right] \\
&= \sum_{i\in\mathcal{G}} P(i|j)\left[\frac{1}{2} - F_{i,j}\left(\frac{m_{i,i}+y'_j}{2}\right)\right] + \sum_{k\in\mathcal{L}} P(k|j)\left[F_{j,k}\left(\frac{y'_j+m_{k,k}}{2}\right) - \frac{1}{2}\right],
\end{aligned}$$

which is non-positive as long as

$$\sum_{i\in\mathcal{G}} P(i|j)\left[\frac{1}{2} - F_{i,j}\left(\frac{m_{i,i}+y'_j}{2}\right)\right] \;\leq\; \sum_{k\in\mathcal{L}} P(k|j)\left[\frac{1}{2} - F_{j,k}\left(\frac{y'_j+m_{k,k}}{2}\right)\right]. \tag{5}$$

Let i^* minimize $F_{i,j}\left(\frac{m_{i,i}+y'_j}{2}\right)$ over $\mathcal{G}$, and let k^* maximize $F_{j,k}\left(\frac{m_{i,i}+y'_j}{2}\right)$ over $\mathcal{L}$. Then inequality (5) holds if

$$\left[\frac{1}{2} - F_{i^*,j}\left(\frac{m_{i^*,i^*}+y'_j}{2}\right)\right]\sum_{i\in\mathcal{G}} P(i|j) \;\leq\; \left[\frac{1}{2} - F_{j,k^*}\left(\frac{y'_j+m_{k^*,k^*}}{2}\right)\right]\sum_{k\in\mathcal{L}} P(k|j). \tag{6}$$

Note that, by (C4), we have $\mathcal{G} \subseteq \{i \in I \mid i \prec j\}$ and $\mathcal{L} = \{i \in I \mid j \precsim i\}$, so (C7) implies $\sum_{i\in\mathcal{G}} P(i|j) \leq \sum_{i\in\mathcal{L}} P(i|j)$. Furthermore, $i^* \precsim j \precsim k^*$ and $y'_j \in [m_{i^*,i^*}, m_{k^*,k^*}]$, so (C6) implies $F_{i^*,j}\left(\frac{m_{i^*,i^*}+y'_j}{2}\right) \geq F_{j,k^*}\left(\frac{y'_j+m_{k^*,k^*}}{2}\right)$. Thus, inequality (6) holds, and it is unprofitable for B to deviate to $y'_j < m_{j,j}$. A symmetric argument applies for deviations $y'_j > m_{j,j}$. ∎

We have shown that condition (C7) on the signals' conditional correlation together with regularity conditions (C5) and (C6) ensure the existence of a pure strategy equilibrium. We now establish that, under (C5) and a strengthening of (C6), condition (C7) is actually necessary for equilibrium existence. Condition (C6′) strengthens (C6) by stating it with equality.

(C6′) For all signals $i, j, k \in I$ with $i \precsim j \precsim k$,

$$F_{i,j}\left(\frac{m_{i,i}+z}{2}\right) = F_{j,k}\left(\frac{m_{k,k}+z}{2}\right)$$

for all $z \in [m_{i,i}, m_{k,k}]$.

It is immediate that, under (C5), condition (C6′) is satisfied in the Shape-Invariant Model. To see this, take any signals i, j, and note that by (C5) we have

$$\frac{m_{i,i}+z}{2} = m_{i,j} + \frac{z-m_{j,j}}{2} \quad \text{and} \quad \frac{m_{k,k}+z}{2} = m_{j,k} + \frac{z-m_{j,j}}{2}.$$

Thus,

$$F_{i,j}\left(\frac{m_{i,i}+z}{2}\right) = F_{i,j}\left(m_{i,j} + \frac{z-m_{j,j}}{2}\right) = F_{j,k}\left(m_{j,k} + \frac{z-m_{j,j}}{2}\right) = F_{j,k}\left(\frac{m_{k,k}+z}{2}\right),$$

where the second equality uses shape-invariance. Under (C5), condition (C6′) also holds in the Stacked-Uniform Model for $a \geq b_N - b_1$: To see this, take i, j, k as in (C6′) and $z \in [m_{i,i}, m_{k,k}]$; then, because $F_{i,j}$ is linear with slope $\frac{a}{2}$ over $[m_{i,i}, m_{k,k}]$, we have

$$F_{i,j}\left(\frac{m_{i,i}+z}{2}\right) = F_{i,j}\left(m_{i,j} + \frac{z-m_{j,j}}{2}\right) = \frac{1}{2} + \frac{z-m_{j,j}}{4a},$$

and similarly for $F_{j,k}\left(\frac{m_{k,k}+z}{2}\right)$.

Theorem 4 *In the Canonical Model, given conditions (C5) and (C6′), condition (C7) is necessary and sufficient for the existence of the pure strategy Bayesian equilibrium. In that equilibrium, candidates locate at $m_{i,i}$, following signal $i \in I$.*

The proof follows the proof of Theorem 3 and is omitted. Since (C7) is difficult to sustain when the number of possible signals is large, Theorem 4 points to the importance of mixed strategies in the electoral model.

5 Existence of Mixed Strategy Equilibria

Our results for pure strategy equilibria suggest that if there are many signals, then pure strategy equilibria may well fail to exist. We now consider mixed strategy equilibria in the electoral game. We let candidate A randomize over campaign platforms following signal s according to a distribution G_s. A mixed strategy for A is a vector $G = (G_s)$ of such distributions, and a mixed strategy for B is a vector $H = (H_t)$. We follow the above convention and let $G_s(z)^-$ and $H_t(z)^-$ be the left-hand limits of these distributions, e.g., $G_s(z)^- = \lim_{w \uparrow z} G_s(w)$. Accordingly, G_s has an atom at x if and only if $G_s(x) - G_s(x)^- > 0$.

To extend our definition of interim expected payoffs, we denote the probability that A wins using platform x following signal s when B uses platform y following signal t as

$$\pi_A(x, y|s, t) \;=\; \begin{cases} F_{s,t}\left(\frac{x+y}{2}\right) & \text{if } x < y \\ 1 - F_{s,t}\left(\frac{x+y}{2}\right) & \text{if } y < x \\ \frac{1}{2} & \text{if } x = y, \end{cases}$$

and we let $\pi_B(\cdot|s, t) = 1 - \pi_A(\cdot|s, t)$. Then, given mixed strategies (G, H), candidate A's interim expected payoff conditional on signal s is

$$\Pi_A(G, H|s) \;=\; \sum_{t \in T} P(t|s) \int \pi_A(x, y|s, t) G_s(dx) H_t(dy),$$

and B's interim payoff $\Pi_B(G, H|t)$ is defined analogously. Abusing notation slightly, let $\Pi_A(X, H|s)$ be A's expected payoff from the degenerate mixed strategy with $G_s(x_s) - G_s(x_s)^- = 1$ for all $s \in S$, and let $\Pi_B(G, Y|t)$ be the analogous expected payoff for B. A *mixed strategy Bayesian equilibrium* is a strategy pair (G, H) such that

$$\Pi_A(G, H|s) \;\geq\; \Pi_A(G', H|s),$$

for all signals $s \in S$ and all strategies G', and

$$\Pi_B(G, H|t) \;\geq\; \Pi_B(G, H'|t),$$

for all signals $t \in T$ and all strategies H'.

Note that X can be a discontinuity point of $\Pi_A(\cdot, H|s)$ only if H_t puts positive probability on a point y such that $\pi_A(\cdot, y|s, t)$ is discontinuous at x_s for some $t \in T$. By continuity of the conditional distributions, there is only one such point, namely, $y = x_s$. Therefore, since each H_t can have at most a countable number of atoms, candidate A's expected payoff function is continuous on all but perhaps a countable set of pure strategies. Furthermore, in equilibrium, if X is a continuity point of $\Pi_A(\cdot, H|s)$ in the support of G_s, then the expected payoff from X conditional on signal s must be $\Pi_A(G, H|s)$. Candidate A must therefore be indifferent over all such points.

As with pure strategies, we can define *ex ante* expected payoffs as

$$\Pi_A(G, H) \;=\; \sum_{s \in S} P(s) \Pi_A(G, H|s) \quad \text{and} \quad \Pi_B(G, H) \;=\; \sum_{t \in T} P(t) \Pi_B(G, H|t).$$

Thus, mixed strategy Bayesian equilibria of the electoral game are equilibria of a two-player, constant-sum game. In the canonical model, the game is symmetric and we define a *symmetric* mixed strategy Bayesian equilibrium as an equilibrium pair (G, H) of strategies with $G = H$.

The next theorem provides a general existence result for mixed strategy equilibria in which candidates use mixed strategies with supports bounded as follows. Let $\overline{m} = \max\{m_{s,t} : s \in S, t \in T\}$ and $\underline{m} = \min\{m_{s,t} : s \in S, t \in T\}$. The interval defined by these "extreme" conditional medians is $M = [\underline{m}, \overline{m}]$. We say (G, H) *has support in* M if the candidates put probability one on M following all signal realizations: For all $s \in S$, $G_s(\overline{m}) - G_s(\underline{m})^- = 1$; and for all $t \in T$, $H_t(\overline{m}) - H_t(\underline{m})^- = 1$.

Theorem 5 *There exists a mixed strategy Bayesian equilibrium with support in M. Under (C1) and (C2), there exists a symmetric mixed strategy Bayesian equilibrium with support in M.*

Proof: We use the existence theorem of Dasgupta and Maskin (1986) for multi-player games with one-dimensional strategy spaces. To apply this result, view the electoral game as a $|S|+|T|$-player game in which each type (corresponding to different signal realizations) of each candidate is a separate player. Player s (or t) has strategy space $M \subseteq \Re$, a compact and convex set, with pure strategies x_s (or y_t). Then $(X, Y) = (x_s, y_t)_{s\in S,t\in T}$ is a pure strategy *profile,* one for each type. Let (X_{-s}, Y) denote the result of deleting x_s from (X, Y). The payoff function of player $s \in S$ is

$$U_s(X, Y) = P(s)\Pi_A(X, Y|s),$$

and the payoff function of player $t \in T$ is

$$U_t(X, Y) = P(t)\Pi_B(X, Y|t).$$

The space of mixed strategies for each player type s (or t) is $\mathcal{M}$, the Borel probability measures on M, with mixed strategies denoted G_s (or H_t). Then (G, H) is a mixed strategy *profile,* one for each type. Note that

$$\sum_{s\in S} U_s(X, Y) + \sum_{t\in T} U_t(X, Y) = 1,$$

for all X and Y, so that the total payoff is trivially upper semi-continuous. Furthermore, payoffs are between zero and one, so they are bounded. Note that U_s is discontinuous at (X, Y) only if $x_s = y_t$ for some $t \in T$. Therefore, the discontinuity points of U_s lie in a set that can be written as $A^*(s)$, as in Dasgupta and Maskin's equation (2). The discontinuity points of U_t lie in a similar set. It remains to show that U_s (likewise U_t) is weakly lower semi-continuous in x_s; that is, for all $x_s \in M$, there exists a $\lambda \in [0,1]$ such that for all (X_{-s}, Y),

$$U_s(X,Y) \;\leq\; \lambda \liminf_{z\downarrow x_s} U_s(z, X_{-s}, Y) + (1-\lambda) \liminf_{z\uparrow x_s} U_s(z, X_{-s}, Y).$$

It is straightforward to verify that this condition holds with equality for $\lambda = \frac{1}{2}$. Let $T^+ = \{t \in T : x_s < y_t\}$, let $T^- = \{t \in T : y_t < x_s\}$, and let $T^0 = \{t \in T : x_s = y_t\}$. Since

$$\begin{aligned} & U_s(X,Y) \\ &= \sum_{t\in T^-} P(s,t)\left(1 - F_{s,t}\left(\frac{x_s+y_t}{2}\right)\right) + \sum_{t\in T^0} \frac{P(s,t)}{2} + \sum_{t\in T^+} P(s,t) F_{s,t}\left(\frac{x_s+y_t}{2}\right), \end{aligned}$$

it follows that

$$\liminf_{z\uparrow x_s} U_s(z, X_{-s}, Y) \;=\; \sum_{t\in T^-} P(s,t)\left(1 - F_{s,t}\left(\frac{x_s+y_t}{2}\right)\right) + \sum_{t\in T^0\cup T^+} P(s,t) F_{s,t}\left(\frac{x_s+y_t}{2}\right)$$

and

$$\liminf_{z\downarrow x_s} U_s(z, X_{-s}, Y) \;=\; \sum_{t\in T^-\cup T^0} P(s,t)\left(1 - F_{s,t}\left(\frac{x_s+y_t}{2}\right)\right) + \sum_{t\in T^+} P(s,t) F_{s,t}\left(\frac{x_s+y_t}{2}\right).$$

The claim

$$U_s(X,Y) \;=\; \frac{1}{2}\liminf_{z\downarrow x_s} U_s(z, X_{-s}, Y) + \frac{1}{2}\liminf_{z\uparrow x_s} U_s(z, X_{-s}, Y)$$

then follows immediately. The same argument can be used to verify that U_t is weakly lower semi-continuous. Then by Dasgupta and Maskin's (1986) Theorem 5, there exists a mixed strategy equilibrium of the multi-player game, and therefore of the electoral game when strategies are restricted to M. To see that following a signal s, candidate A has no profitable deviations outside M, take any $x > \overline{m}$. Note that for all $t \in T$ and all $y \in M$, we have $\pi_A(\overline{m}, y|s,t) \geq \pi_A(x, y|s,t)$. Let G' be any deviation such that G'_s puts probability one on $x_s > \overline{m}$, and let G'' put probability one on $\overline{m}$ instead. Then

$$\Pi_A(G', H|s) \;\leq\; \Pi_A(G'', H|s) \;\leq\; \Pi_A(G, H|s).$$

A similar argument applies when $x < \underline{m}$, yielding the claim. Adding (C1) and (C2), we see that the electoral game is a two-player, symmetric constant-sum game. Therefore, by existence of equilibrium and by interchangeability, there exists a symmetric mixed strategy equilibrium. ∎

The proof of existence of a mixed strategy equilibrium in Theorem 5 relies on the assumption of continuous conditional distributions. If discontinuities are allowed for, then the weak lower semicontinuity condition of Dasgupta and Maskin (1986) may be violated. A weaker sufficient condition for existence in symmetric games that might be applied is Reny's (1999) diagonal better reply security. The next example shows, however, that Reny's condition can be violated even if only the very restricted discontinuities of the Generalized Downsian Model are present. Thus, the prospects for a more general result using known sufficient conditions for existence in discontinuous games seem poor.

Example 2 *Diagonal Better Reply Security Violated in the Generalized Downsian Model.* Consider a discontinuous version of the Canonical Model in which $I = \{1, 2, 3\}$ and, for all $i, j \in I$, $F_{i,j}$ is the point mass on $m_{i,j}$, given in the table below. We assign priors on $I \times I$ as indicated there.

$j = 3$	$P(1,3) = .16\epsilon$ $m_{1,3} = 1.8$	$P(2,3) = .41\epsilon$ $m_{2,3} = 2.3$	$P(3,3) = 1 - 2.89\epsilon$ $m_{3,3} = 2.6$
$j = 2$	$P(1,2) = .45\epsilon$ $m_{1,2} = 1$	$P(2,2) = .05\epsilon$ $m_{2,2} = 1.9$	$P(3,2) = .41\epsilon$ $m_{3,2} = 2.3$
$j = 1$	$P(1,1) = .8\epsilon$ $m_{1,1} = 0$	$P(2,1) = .45\epsilon$ $m_{2,1} = 1$	$P(3,1) = .16\epsilon$ $m_{3,1} = 1.8$
	$i = 1$	$i = 2$	$i = 3$

Define the mixed strategy G as follows: $G_1(m_{1,1}) - G_1(m_{1,1})^- = \alpha$, $G_1(m_{1,2}) - G_1(m_{1,2})^- = 1 - \alpha$, $G_2(2) - G_2(2)^- = 1$, and $G_3(m_{3,3}) - G_3(m_{3,3})^- = 1$, and set $\alpha = .8$ and $H = G$. That is, after signal 1, the candidates mix between two conditional medians, $m_{1,1}$ and $m_{1,2}$; after signal 2, the candidates adopt the platform 2 (which does not correspond to a conditional median); and after signal 3, the candidates adopt the conditional median $m_{3,3}$. Reny's (1999) diagonal better reply security requires that, if (G, H) is not a mixed strategy Bayesian equilibrium, then candidate A has a mixed strategy deviation that is profitable, even if B's mixed strategy is allowed to vary within some open set. Specifically, there must exist a mixed strategy $\hat{G}$ for A and an open set $\mathcal{H}$ of mixed strategies for B such that $H \in \mathcal{H}$ and $\inf_{\hat{H} \in \mathcal{H}} \Pi_A(\hat{G}, \hat{H}) > \frac{1}{2}$.[6]

Note that this strategy is clearly a best response following signal 3, if we set $\epsilon > 0$ sufficiently

[6]Here, we give candidate B's strategy space the product topology, where each factor, the set of distributions over the real line, is given the weak* topology.

small. Following signal 2, candidate A would tie with candidate B in case B received signal 1 and positioned at $m_{1,1} = 0$, would lose to B in case B received signal 1 and positioned at 1, and would tie with B in case B received signals 2 or 3: A's expected payoff would be

$$\begin{aligned} P(2)\Pi_A(G, H|2) &= \alpha P(1,2)(.5) + (1-\alpha)P(1,2)(0) + P(2,2)(.5) + P(2,3)(.5) \\ &= \epsilon(.225\alpha + .23) \;=\; .41\epsilon, \end{aligned}$$

where we weight the interim payoff by the marginal probability of signal 2. If candidate A moved to the left following signal 2, the candidate could move a small enough amount to $x'_2 \in (1.8, 2)$ to win against candidate B in case B received signal 1 and positioned at $m_{1,1} = 0$ or received signal 2, but would then lose against B in case B received signal 3: A's expected payoff would again be

$$\begin{aligned} P(2)\Pi_A(X', H|2) &= \alpha P(1,2)(1) + (1-\alpha)P(1,2)(0) + P(2,2)(1) + P(2,3)(0) \\ &= \epsilon(.45\alpha + .05) \;=\; .41\epsilon. \end{aligned}$$

Moving further to the left, A could do no better than locate at $m_{1,2} = 1$, which yields an expected payoff of

$$\begin{aligned} P(2)\Pi_A(X', H|2) &= \alpha P(1,2)(1) + (1-\alpha)P(1,2)(.5) + P(2,2)(0) + P(2,3)(0) \\ &= \epsilon(.225\alpha + .225) \;=\; .405\epsilon. \end{aligned}$$

Moving to the right, the best A could do would be to win against B in case B received signal 3 and lose otherwise, which yields an expected payoff of

$$P(2)\Pi_A(X', H|2) \;=\; \alpha P(1,2)(0) + (1-\alpha)P(1,2)(0) + P(2,2)(0) + P(2,3)(1) \;=\; .41\epsilon.$$

Thus, G is a best response to H following signal 2 as well.

Define G' as G, but with $G'_1(m_{1,1}) - G'_1(m_{1,1})^- = 1$; that is, according to G', candidate A plays as in G but chooses the conditional median $m_{1,1}$ with probability one following signal 1. Letting X be the pure strategy with $x_1 = m_{1,2}$, we have

$$\begin{aligned} P(1)\Pi_A(X, H|1) &= P(1,1)(\alpha(0) + (1-\alpha)(.5)) + P(1,2)(1) + P(1,3)(.5) \\ &= \epsilon(.1\alpha + .53) \;=\; .61\epsilon \end{aligned}$$

and

$$\begin{aligned} P(1)\Pi_A(G', H|1) &= P(1,1)(\alpha(.5) + (1-\alpha)(1)) + P(1,2)(.5) + P(1,3)(0) \\ &= \epsilon(.6\alpha + .225) \;=\; .705\epsilon. \end{aligned}$$

Since G_1 puts positive probability on $x_1 = m_{1,2}$, we conclude that (G, H) is not a Bayesian equilibrium. Thus, diagonal better reply security requires $\hat{G}$ and $\mathcal{H}$, as described above. Note that $H \in \mathcal{H}$, and that G_2 and G_3 are best responses to H conditional on signals 2 and 3, respectively.

Furthermore, we claim that the only position that increases A's expected payoff conditional on signal 1 is, in fact, $x_1 = m_{1,1}$. If A took a position $x_1' \in (m_{1,1}, m_{1,2})$ following signal 1, then A's expected payoff would be

$$\begin{aligned} P(1)\Pi_A(X', H|1) &= P(1,1)(\alpha(0) + (1-\alpha)(1)) + P(1,2)(1) + P(1,3)(0) \\ &= \epsilon(1.25 - .8\alpha) \;=\; .61\epsilon. \end{aligned}$$

If A took a position $x_1' \in (m_{1,2}, 2)$ following signal 1, then A's expected payoff would again be

$$\begin{aligned} P(1)\Pi_A(X', H|1) &= P(1,1)(\alpha(0) + (1-\alpha)(0)) + P(1,2)(1) + P(1,3)(1) \\ &= .61\epsilon. \end{aligned}$$

And positioning further to the right would yield an even lower expected payoff. Therefore, $\hat{G}$ must involve the transfer of probability mass from $m_{1,2}$ to $m_{1,1}$ following signal 1.

The difficulty for diagonal better reply security is that such a change no longer delivers an *ex ante* expected payoff for A above one half if we perturb H slightly to $\hat{H}$ by specifying that $\hat{H}_2$ put probability one on a point to the left of, and close to, 2: In that case, A loses to B when A receives signal 1 and locates at $m_{1,1}$ and B receives signal 2. To see the claim, consider $\hat{G} = G'$. Then A's *ex ante* expected payoff in excess of one half, $\Pi_A(\hat{G}, \hat{H}) - \frac{1}{2}$, is

$$\begin{aligned} &P(1)\Pi_A(G', \hat{H}|1) + P(2)\Pi_A(G', \hat{H}|2) + P(3)\Pi_A(G', \hat{H}|3) - .5 \\ &= [P(1,1)(\alpha(.5) + (1-\alpha)(1)) + P(1,2)(0) + P(1,3)(0)] \\ &\quad + [P(1,2)(\alpha(.5) + (1-\alpha)(0)) + P(2,2)(0) + P(2,3)(.5)] \\ &\quad + [P(1,3)(\alpha(1) + (1-\alpha)(.5)) + P(2,3)(1) + P(3,3)(.5)] - .5 \\ &= \alpha[(.5)(.8\epsilon) - .8\epsilon + (.5)(.45\epsilon) + .16\epsilon - (.5)(.16\epsilon)] \\ &\quad + .8\epsilon + (.5)(.41\epsilon) + (.5)(.16\epsilon) + (.41\epsilon) + (.5)(1 - 2.89\epsilon) - .5 \\ &= \epsilon[\alpha(-.095) + .05] \\ &= -.026\epsilon, \end{aligned}$$

which is negative. Thus, diagonal better reply security is not fulfilled by $\hat{G} = G'$.

Reny's diagonal better reply security condition does not require that $\hat{G} = G'$, as in the above calculation: Candidate A could, for example, move probability mass from 2 or $m_{3,3}$ following signals 2 and 3, respectively; as already confirmed, this would not increase A's *ex ante* expected payoff when B uses H, but it could conceivably mitigate the problem illustrated in the preceding, "protecting" A from B's slight move to the left following signal 2. A closer look shows, however, that no such protection is available. Following signal 3, of course, any change in G'_3 will lead to a discontinuous decrease in A's expected payoff, weighted by $1-2.89\epsilon$, which can be made arbitrarily close to one. Following signal 2, A might move probability mass from 2 to the left to defeat candidate B in case B also receives signal 2, but such a change means that A would lose to B in case B received signal 3, and we have seen that the two effects cancel. Finally, after signal 1, A might move probability mass from $m_{1,1}$ to the right in order to defeat B when B receives signal 2, but we have seen that such a move does not increase A's interim expected payoff conditional on signal 1. This completes the example. □

An alternative approach to equilibrium existence in the Generalized Downsian Model would be to restrict first the strategies of the candidates to the set $\{m_{s,t} \mid s \in S, t \in T\}$ of conditional medians, find an equilibrium of the restricted game, and then prove that these strategies form an equilibrium of the unrestricted game. While this approach seems plausible, example 5 demonstrates that it does not work.

Example 3 *An Equilibrium of the Restricted Generalized Downsian Model Is Not an Equilibrium of the Unrestricted Game.* Let $I = \{1, 2, 3\}$, where each $F_{i,j}$ is a point mass and priors and conditional medians are as below.

$j = 2$	$P(0,2) = 0$	$P(1,2) = \frac{2\epsilon}{5}$ $m_{1,2} = 1.7$	$P(2,2) = \frac{1}{2} - \epsilon$ $m_{2,2} = 2$
$j = 1$	$P(0,1) = \frac{2\epsilon}{5}$ $m_{0,1} = .8$	$P(1,1) = \frac{2\epsilon}{5}$ $m_{1,1} = 1$	$P(2,1) = \frac{2\epsilon}{5}$ $m_{2,1} = 1.7$
$j = 0$	$P(0,0) = \frac{1}{2} - \epsilon$ $m_{0,0} = 0$	$P(1,0) = \frac{2\epsilon}{5}$ $m_{1,0} = .8$	$P(2,0) = 0$
	$i = 0$	$i = 1$	$i = 2$

Define the pure strategy X by $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$, let $Y = X$, and note that (X, Y) is a pure strategy Bayesian equilibrium of the electoral game restricted to the conditional medians $\{0, .8, 1, 1.7, 2\}$. This strategy is clearly a best response following signals 0 and 2. Following signal 1, playing $x_1 = 1$ produces a win when B receives signal 0, a tie when B receives signal 1, and a loss when B receives signal 2. Moving to $x_1' = 1.7$, for example, produces a tie, a loss, and a win, respectively. Since these outcomes have equal weight, A's expected payoff is maximized at $x_1 = 1$ over the conditional medians. In the unrestricted game, however, $x_1' = 1.5$ yields a higher expected payoff of $\left(\frac{1}{3}\right)(1) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)(1) > \frac{1}{2}$. □

We now study the continuity properties of the mixed strategy equilibrium correspondence as we vary the parameters of the model, specifically the candidates' marginal prior on $S \times T$ and the conditional distributions of μ. We index specifications of our model by γ, where the marginal probability of (s,t) in game γ is $P^{\gamma}(s,t)$, and the distribution of μ conditional on s and t in γ is $F^{\gamma}_{s,t}$. To consider continuity properties, we assume γ lies in a metric space Γ, and that the indexing is continuous: For each $s \in S$ and $t \in T$, if $\gamma_n \to \gamma$, then $P^{\gamma_n}(s,t) \to P^{\gamma}(s,t)$ and $F^{\gamma_n}_{s,t} \to F^{\gamma}_{s,t}$ weakly. Denote the interval defined by the extreme conditional medians in game γ by $M(\gamma)$, and note that by the assumption of continuous indexing, the correspondence $M \colon \Gamma \rightrightarrows \Re$ so-defined is continuous.

Theorem 5 establishes the existence of a mixed strategy equilibrium for all $\gamma \in \Gamma$. Since the electoral game is constant-sum, the *ex ante* expected payoff of a candidate in game γ is the same in all mixed strategy equilibria. Denote these payoffs or "values" by $v_A(\gamma)$ and $v_B(\gamma)$. Furthermore, each candidate has an "optimal" mixed strategy that guarantees the candidate's value, no matter which strategy the opponent uses. If (C1) and (C2) hold for game γ, then the game is symmetric, so that $v_A(\gamma) = v_B(\gamma) = \frac{1}{2}$. The next theorem establishes that $v_A(\gamma)$ and $v_B(\gamma)$ vary continuously in the parameters of the game, even when asymmetries are allowed.

Theorem 6 *The mapping $v_A \colon \Gamma \to \Re$ is continuous.*

Proof: We prove lower semi-continuity of v_A at γ. A symmetric argument proves lower semi-continuity of $v_B = 1 - v_A$, which, in turn, gives us upper semi-continuity of v_A. Let $\gamma_n \to \gamma$, and suppose $v_A(\gamma) > \liminf v_A(\gamma_n)$. Let Π^n_A denote A's *ex ante* expected payoff function corresponding to γ_n, and let Π_A denote the *ex ante* payoffs corresponding to γ. Let M^n denote the interval $M(\gamma^n)$, let $M = M(\gamma)$, and let $\hat{M}$ be any compact set containing M in its interior. By continuity, therefore, $M^n \subseteq \hat{M}$ for high enough n. For each n, let (G^n, H^n) be an equilibrium with support in M^n for the electoral game indexed by γ_n, so $\Pi^n_A(G^n, H^n) = v_A(\gamma_n)$ and

$\Pi^n_B(G^n, H^n) = v_B(\gamma_n)$. By compactness of $\hat{M}$, there exists a weakly convergent subsequence of $\{(G^n, H^n)\}$, also indexed by n, with limit (G, H). Going to a further subsequence if necessary, we may assume $\{v_A(\gamma_n)\}$ converges to limit $v < v_A(\gamma)$. Let (G^*, H^*) be an equilibrium of the electoral game indexed by γ, so G^* is an optimal strategy for A, which guarantees a payoff of at least $v_A(\gamma)$ in game γ. Thus, $\Pi_A(G^*, H) \geq v_A(\gamma)$. In particular, there exists a pure strategy X^* such that

$$\Pi_A(X^*, H) \;\geq\; v_A(\gamma) \;>\; v.$$

We claim that, as a consequence, there exists a pure strategy X' such that

$$\Pi^n_A(X', H^n) \quad > \quad \frac{\Pi_A(X^*, H) + v}{2},$$

for high enough n. But this, with $v_A(\gamma_n) \to v$, contradicts the assumption that G^n is a best response to H^n for candidate A. We establish the claim in three steps.

Step 1. By Lemma 1 (in the appendix), for every $s \in S$, either

$$\sum_{t\in T} P(t|s)[H_t(x^*_s) - H_t(x^*_s)^-][F_{s,t}(x^*_s)] \quad \geq \quad \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(x^*_s) - H_t(x^*_s)^-] \tag{7}$$

or

$$\sum_{t\in T} P(t|s)[H_t(x^*_s) - H_t(x^*_s)^-][1 - F_{s,t}(x^*_s)] \quad \geq \quad \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(x^*_s) - H_t(x^*_s)^-]. \tag{8}$$

Let S^- be the set of $s \in S$ such that (7) holds, and let S^+ be the set of $s \in S \setminus S^-$ such that (8) holds. For $s \in S^-$, let $\{x^k_s\}$ be a sequence increasing to x^*_s, and for $s \in S^+$, let $\{x^k_s\}$ be a sequence decreasing to x^*_s. In addition, we choose each x^k_s to be a continuity point of H_t for all $t \in T$; this is possible because T is finite and each H_t has a countable number of discontinuity points. Thus, $H_t(x^k_s) - H_t(x^k_s)^- = 0$ for all $t \in T$. For each k, define the strategy $X^k = (x^k_s)$ for candidate A.

Step 2. We now argue that X^k satisfies $\liminf \Pi_A(X^k, H) \geq \Pi_A(X^*, H)$. For each $t \in T$, let λ_t denote the probability measure generated by the distribution H_t, let μ_t denote the degenerate measure with mass $H_t(x^*_s) - H_t(x^*_s)^-$ on each x^*_s, and let $\nu_t = \lambda_t - \mu_t$. Let

$$\pi^*_{s,t}(z) = \pi_A(x^*_s, z|s, t)$$

denote A's probability of winning using x^*_s conditional on signal s when B receives signal t and chooses platform z, and let

$$\pi^k_{s,t}(z) = \pi_A(x^k_s, z|s, t)$$

denote A's analogous probability of winning using x_s^k. Note that

$$\begin{aligned}\Pi_A(X^k,H) \quad - \quad \Pi_A(X^*,H) = \sum_{s\in S} P(s) \sum_{t\in T} P(t|s) \int [\pi_{s,t}^k(z) - \pi_{s,t}^*(z)]\, \lambda_t(dz) \\ = \quad \sum_{s\in S^-}\sum_{t\in T} P(s,t)[H_t(x_s^*) - H_t(x_s^*)^-]\left[F_{s,t}\left(\frac{x_s^*+x_s^k}{2}\right) - \frac{1}{2}\right] \\ + \sum_{s\in S^+}\sum_{t\in T} P(s,t)[H_t(x_s^*) - H_t(x_s^*)^-]\left[1 - F_{s,t}\left(\frac{x_s^*+x_s^k}{2}\right) - \frac{1}{2}\right] \\ + \sum_{s\in S}\sum_{t\in T} P(s,t) \int [\pi_{s,t}^k(z) - \pi_{s,t}^*(z)]\, \nu_t(dz).\end{aligned}$$

Since $\pi_{s,t}^k - \pi_{s,t}^* \to 0$ almost everywhere (ν_t), the corresponding integral terms above converge to zero. Thus, by construction of X^k,

$$\lim \inf_{m\to\infty} \Pi_A(X^k,H) \;\geq\; \Pi_A(X^*,H) \;>\; v,$$

as desired.

Step 3. Choose k such that $\Pi_A(X^k,H) > \frac{\Pi_A(X^*,H)+v}{2}$ and set $X' = X^k$. To prove the claim that $\Pi_A^n(X',H^n) > v$ for high enough n, define the functions

$$\phi'_{s,t}(z) = \pi_A(x'_s, z|s,t) \;\text{ and }\; \phi^n_{s,t}(z) = \pi_A^{\gamma_n}(x'_s, z|s,t).$$

Note that

$$\Pi_A(X',H) \quad = \quad \sum_{s\in S}\sum_{t\in T} P(s,t) \int \phi'_{s,t}(z)\, H_t(dz),$$

and, letting $P^n = P^{\gamma_n}$,

$$\Pi_A^n(X',H^n) \quad = \quad \sum_{s\in S}\sum_{t\in T} P^n(s,t) \int \phi^n_{s,t}(z)\, H^n_t(dz).$$

Since $\Pi_A(X',H) > v$, it suffices to show that

$$\int \phi^n_{s,t}(z)\, dH^n_t \to \int \phi'_{s,t}(z)\, dH_t,$$

for each $s \in S$ and $t \in T$. To prove this, fix $\epsilon > 0$. Because $x'_s = x_s^k$ is not a mass point of H_t, we may specify an interval $Z = [\underline{z}, \overline{z}]$ with $x'_s \in (\underline{z}, \overline{z})$ such that $H_t(\overline{z}) - H_t(\underline{z})^- < \frac{\epsilon}{4}$. By weak convergence, $H^n_t(\overline{z}) - H^n_t(\underline{z})^- < \frac{\epsilon}{2}$ for sufficiently high n. Furthermore, $\{\phi^n_{s,t}\}$ is a sequence of functions that are non-decreasing on $[\underline{m}, \underline{z}]$ and converge pointwise to $\phi'_{s,t}$ on this interval, so they converge uniformly to $\phi'_{s,t}$ on the interval. Similarly, each $\phi^n_{s,t}$ is non-increasing on

$[\overline{z}, \overline{m}]$, so the functions converge uniformly to $\phi'_{s,t}$ on this interval. Choosing n high enough that $|\phi^n_{s,t}(z) - \phi'_{s,t}(z)| < \frac{\epsilon}{2}$ for all $z \in [\underline{m}, \underline{z}] \cup [\overline{z}, \overline{m}]$, yields

$$\left| \int \phi^n_{s,t}(z)\, dH^n_t - \int \phi'_{s,t}(z)\, dH_t \right| \quad < \quad \epsilon,$$

as required. ∎

Letting $\mathcal{B}$ denote the Borel probability measures over X, define the mixed strategy equilibrium correspondence $E\colon \Gamma \rightrightarrows \mathcal{B}^{S \cup T}$ so that $E(\gamma)$ consists of all mixed strategy equilibrium pairs (G, H). We have shown that this correspondence has non-empty values. The next result establishes an important continuity property of the equilibrium correspondence.

Theorem 7 *The correspondence $E\colon \Gamma \rightrightarrows \mathcal{B}^{S \cup T}$ has closed graph.*

Proof: Since Γ and $\mathcal{B}^{S \cup T}$ are metrizable, we may restrict attention to sequences, rather than nets. Let $\gamma_n \to \gamma$, let $(G^n, H^n) \in E(\gamma_n)$ for each n, and suppose that $(G^n, H^n) \to (G, H)$. If $(G, H) \notin E(\gamma)$, then, using the notation from the proof of Theorem 6, one candidate, say A, has a pure strategy X such that

$$\Pi_A(X, H) \quad > \quad v_A(\gamma). \tag{9}$$

But then, as in the proof of Theorem 6, we can find a strategy X' satisfying (9) such that no x'_s is a mass point of any H_t, and then we can show that

$$\Pi^n_A(X', H^n) \quad > \quad \frac{\Pi_A(X, H) + v_A(\gamma)}{2},$$

for high enough n. But $v_A(\gamma_n) \to v_A(\gamma)$ by Theorem 6, so it follows that

$$\Pi^n_A(X', H^n) \quad > \quad v_A(\gamma_n),$$

for high enough n, contradicting the assumption that G^n is a best response to H^n for A in the electoral game indexed by γ_n. ∎

Despite the discontinuities present in the electoral game, Theorem 7 delivers a desirable robustness property for mixed strategy equilibria: If we perturb the game slightly, then mixed strategy equilibria cannot be far away from the mixed strategy equilibria for the original specification of the game. Because we assume each model $\gamma \in \Gamma$ satisfies our maintained assumption of continuous conditional distributions, however, this continuity result does not apply directly to the issue of robustness of the unique pure strategy equilibrium in the Downsian Model. In fact, continuity is critical for the arguments above, and we use a separate line of argument in the next section to extend our continuity result.

6 Robustness of the Median Voter Theorem

To consider the robustness of the Downsian equilibrium, we say that a sequence $\{\gamma_n\}$ in Γ is *approximately Downsian* if: (i) for all $s \in S$ and all $t \in T$, $F_{s,t} = \lim F_{s,t}^{\gamma_n}$ is degenerate on some $m_{s,t} \in \Re$, and letting $P = \lim P^{\gamma_n}$, (ii) for all $s \in S$ and all $t \in T$, if $P(s,t) > 0$, then $P(t|s) = P(s|t) = 1$. Note the implicit requirement that $\{F_{s,t}^{\gamma_n}\}$ and $\{P^{\gamma_n}(s,t)\}$ indeed converge for all $s \in S$ and all $t \in T$. Thus, while the "limiting model" is Downsian and exhibits discontinuous conditional distributions, it is clear that the sequence of models indexed by γ_n gets arbitrarily "close" to Downsian Model. Our first result extends the continuity result of Theorem 6 to the Downsian Models, where the value of the game to each candidate is one half.

Theorem 8 *If the sequence $\{\gamma_n\}$ in Γ is approximately Downsian, then $v_A(\gamma_n) \to \frac{1}{2}$.*

Proof: Let $\{\gamma_n\}$ be approximately Downsian. We will show that $\liminf v_A(\gamma_n) \geq \frac{1}{2}$, and a symmetric argument for candidate B will then imply that $\lim v_A(\gamma_n) = \frac{1}{2}$. For each n, let $F_{s,t}^n$ denote the distribution of μ conditional on signal pair (s,t) and let $P^n(s,t)$ denote the prior probability of signal pair (s,t) in γ_n, and let $m_{s,t}^n$ denote the median of $F_{s,t}^n$. Define the mapping $\tau \colon S \to T$ so that, for each $s \in S$, we have $P(\tau(s)|s) = 1$. That is, in the limiting Downsian Model, A's signal s corresponds to B's signal $\tau(s)$. Now let X^n be defined by $x_s^n = m_{s,\tau(s)}^n$ for all $s \in S$, and let Y^n be an arbitrary pure strategy for B. Then A's expected payoff from (X^n, Y^n) conditional on signal s in game γ^n is

$$
\begin{aligned}
&\Pi_A^n(X^n, Y^n|s) \\
&= \sum_{t \in T: x_s^n < y_t^n} P^n(t|s) F_{s,t}^n\left(\frac{x_s^n + y_t^n}{2}\right) + \sum_{t \in T: y_t^n < x_s^n} P^n(t|s)\left(1 - F_{s,t}^n\left(\frac{x_s^n + y_t^n}{2}\right)\right) \\
&\quad + \frac{1}{2} \sum_{t \in T: x_s^n = y_t^n} P^n(t|s)
\end{aligned}
$$

Note that $P^n(\tau(s)|s) \to 1$ and $P^n(t|s) \to 0$ for $t \neq \tau(s)$, and define the sequence $\{\Phi^n\}$ by

$$
\Phi^n = \begin{cases} F_{s,\tau(s)}^n\left(\frac{m_{s,\tau(s)}^n + y_{\tau(s)}^n}{2}\right) & \text{if } m_{s,\tau(s)}^n < y_{\tau(s)}^n \\ 1 - F_{s,\tau(s)}^n\left(\frac{m_{s,\tau(s)}^n + y_{\tau(s)}^n}{2}\right) & \text{if } y_{\tau(s)}^n < m_{s,\tau(s)}^n \\ \frac{1}{2} & \text{else.} \end{cases}
$$

It follows that

$$
\liminf \Pi_A^n(X^n, Y^n|s) \;\geq\; \liminf \Phi^n \;\geq\; \frac{1}{2},
$$

where the last inequality follows from $\Phi^n \geq \frac{1}{2}$ for all n. Thus, A can guarantee an expected payoff arbitrarily close to one half as n goes to infinity, and we conclude that $\liminf v_A(\gamma_n) \geq \frac{1}{2}$, as required. ∎

Whereas Theorem 6 establishes continuity of the value over the class of models with continuous conditional distribution, Theorem 8 extends this result to certain "boundary points" of this class exhibiting discontinuous conditional distributions, namely, the Downsian Models. A restriction imposed in the latter models is complete information, so that a signal for one candidate uniquely determines the signal for the other candidate. The next example shows that our continuity results do not extend when complete information in the Downsian Model is relaxed: We demonstrate a Generalized Downsian Model at which the value of the game is discontinuous in the model's parameters.

Example 4 *Continuity of Value Violated at Generalized Downsian Model.* Let $I = \{0, 1\}$ and $P(i, j) = \frac{1}{4}$ for all $i, j \in I$, with conditional distributions $F_{i,j}$ equal to the point mass on zero for all $i, j \in I$. For each n, let $F^n_{0,j}$, $j = 0, 1$, be the uniform distribution with density n centered at zero, and let $F^n_{1,j}$, $j = 0, 1$, be the uniform distribution with density n centered at $\frac{1}{n}$. Note that $F^n_{i,j} \to F_{i,j}$ weakly and that the supports of $F^n_{0,j}$ and $F^n_{1,j}$ are contiguous for all $i, j \in I$. Thus, for each n, candidate A has full information about the distribution of the cut point, whereas candidate B receives no information: conditional on $j = 0$ and $j = 1$, the distribution of the cutpoint has mean 0 and mean $\frac{1}{n}$ with equal probability. For each n, define $x^n_0 = 0$, $x^n_1 = \frac{1}{n}$, and $y^n_0 = y^n_1 = 0$, and note that the pure strategy profile (X^n, Y^n) so-defined is a Bayesian equilibrium. In it, candidate A always locates at the median, winning for sure when $i = 1$ and matching B when $i = 0$. Thus, candidate A's equilibrium expected payoff is $\frac{3}{4}$ and the value of the game for A is $v^n_A = \frac{3}{4}$ for all n. In contrast, the Generalized Downsian Model specified above is symmetric, and payoffs for the candidates must be one half in any equilibrium. □

We have shown that the value in models close to Downsian must be close to one half, and the next result pushes this further: In fact, the equilibrium mixed strategy distributions used following any signal realization must themselves be close to the location of the median in the Downsian Model conditional on that signal. Thus, though Example 1 shows that pure strategy equilibria may fail to exist near the Downsian Model, Theorem 5 shows that mixed strategy equilibria do exist, and these equilibria must be close to the Downsian equilibrium.

Theorem 9 *If the sequence $\{\gamma_n\}$ is approximately Downsian and $(G^n, H^n) \in E(\gamma_n)$ for all n, then, for all $s \in S$ and all $t \in T$ with $P(s,t) > 0$, $\{G^n_{s,t}\}$ and $\{H^n_{s,t}\}$ converge weakly to the point mass on $m_{s,t}$.*

Proof: Let $\{\gamma_n\}$ be approximately Downsian, let $(G^n, H^n) \in E(\gamma_n)$ for each n, and suppose H^n_t does not converge to the point mass on m_t for some $t \in T$. Then there exists an interval (a,b) containing m_t such that either $\limsup H^n_t(a) > 0$ or $\liminf H^n_t(b)^- < 1$. Without loss of generality, assume the latter, and consider a subsequence (still indexed by n) along which this lim inf is achieved, i.e., $\lim H^n_t(b)^- < 1$. Using the notation from the proof of Theorem 8, let $s = \tau^{-1}(t)$, and define X^n so that $x^n_{s'} = m^n_{s',t}$ for each $s' \in S$ and each n. As in the proof of Theorem 8, we can show that

$$\liminf_{n\to\infty} \inf_{Y\in\Re^T} \Pi^n_A(X^n, Y|s') \;\; \geq \;\; \frac{1}{2} \tag{10}$$

for all $s' \in S$. Moreover, if we restrict candidate B to strategies such that the candidate locates to the right of b following signal t, this inequality becomes strict conditional on A receiving signal s:

$$\liminf_{n\to\infty} \inf_{Y\in\Re^T\, y_t\geq b} \Pi^n_A(X^n, Y|s) \;\; > \;\; \frac{1}{2}.$$

To see this, let Y^n be an arbitrary pure strategy for B such that $y^n_t \geq b$, and take c such that $m_{s,t} < c < \frac{m_{s,t}+b}{2}$. Note that

$$\liminf \Phi^n \;=\; \liminf\, F^n_{s,t}\left(\frac{m_{s,t}+y_t}{2}\right) \;\geq\; \liminf\, F^n_{s,t}(c)^- \;\geq\; F_{s,t}(c)^- \;>\; \frac{1}{2}, \tag{11}$$

where the second-to-last inequality follows from weak convergence and the strict inequality follows from $c > m_{s,t}$ and our assumption of a unique median. From the above observations, we have

$$\begin{aligned}
&\liminf \Pi^n_A(X^n, H^n|s) \\
&\quad\geq\; \liminf \int_{(-\infty,b)} \pi^n_A(m^n_{s,t}, y|s,t)\, H^n_t(dy) + \liminf \int_{[b,\infty)} \pi^n_A(m^n_{s,t}, y|s,t)\, H^n_t(dy) \\
&\quad\geq\; \frac{1}{2}\lim H^n_t(b)^- + F_{s,t}(c)^-(1 - \lim H^n_t(b)^-),
\end{aligned}$$

which, by $F_{s,t}(c)^- > \frac{1}{2}$ and $\lim H^n_t(b)^- < 1$, is greater than one half. It follows that $\liminf \Pi^n_A(G^n, H^n) > \frac{1}{2}$, i.e., $\liminf v_A(\gamma_n) > \frac{1}{2}$. But $v_A(\gamma_n) \to \frac{1}{2}$ by Theorem 8, a contradiction. ∎

In Example 4, the equilibrium platforms of the candidates converge to zero, the unique equilibrium of the limiting Generalized Downsian Model. This leaves the possibility that, while our continuity result for the value does not extend, the upper hemicontinuity result of Theorem 9 does extend beyond the Downsian Model. The next example shows that this is not so: Theorem 7 cannot be generalized to allow for even the limited discontinuities of the Generalized Downsian Model.

Example 5 *Upper Hemicontinuity Violated at Generalized Downsian Model.* Let $I = \{-1, 0, 1\}$, with uniform priors, i.e., $P(i,j) = \frac{1}{9}$ for each $i, j \in I$. For each $i, j \in I$, let $F_{i,j}$ be the point mass on $m_{i,j}$, and let $\{F^n_{i,j}\}$ be a sequence of uniform distributions with density n, where all conditional medians are depicted below.

	$i=-1$	$i=0$	$i=1$
$j = 1$	$m_{-1,1} = 0$ $m^n_{-1,1} = \frac{1}{n}$	$m_{0,1} = 0$ $m^n_{0,1} = 0$	$m_{1,1} = 0$ $m^n_{1,1} = \frac{1}{n}$
$j = 0$	$m_{-1,0} = 0$ $m^n_{-1,0} = 0$	$m_{0,0} = 0$ $m^n_{0,0} = 0$	$m_{1,0} = 0$ $m^n_{1,0} = 0$
$j = -1$	$m_{-1,-1} = -1$ $m^n_{-1,-1} = -1$	$m_{0,-1} = 0$ $m^n_{0,-1} = 0$	$m_{1,-1} = 0$ $m^n_{1,-1} = \frac{1}{n}$

Thus, each n defines an instance of the Canonical Model, and the conditional distributions $F^n_{1,-1}$, $F^n_{-1,1}$, and $F^n_{1,1}$ converge weakly to the degenerate distributions $F_{1,-1}$, $F_{-1,1}$, and $F_{1,1}$, respectively. Furthermore, the supports of $F^n_{-1,0}$ and $F^n_{-1,1}$ are contiguous, as are the supports of $F^n_{0,-1}$ and $F^n_{1,-1}$. For each n, define the pure strategy X^n as follows: $x^n_{-1} = -1$, $x^n_0 = 0$, and $x^n_1 = \frac{1}{n}$, and let $Y^n = X^n$. Then (X^n, Y^n) is a Bayesian equilibrium of the n^{th} game in the sequence. To see this, note that following signal -1, $x^n_{-1} = -1$ produces a tie if B receives signal -1, a loss if B receives signal 0, and a loss if B receives signal 1, yielding a conditional expected payoff for candidate A of $\frac{1}{6}$. Moving from $x^n_{-1} = -1$ to the right, A's conditional expected payoff is maximized for $x'_{-1} \in [0, \frac{1}{n}]$, which yields

$$\left(\frac{1}{3}\right)(0) + \left(\frac{1}{3}\right) F_{-1,0}\left(\frac{x'_{-1} - x^n_0}{2}\right) + \left(\frac{1}{3}\right)\left(1 - F_{-1,1}\left(\frac{x^n_1 - x'_{-1}}{2}\right)\right) \quad = \quad \frac{1}{6},$$

and thus $x^n_{-1} = -1$ is a best response. Following signal 0, $x^n_0 = 0$ produces a win if B receives signals -1 or 1 and a tie if B receives signal 0, and this is clearly a best response. Following signal 1, $x_1^n = \frac{1}{n}$ produces a win if B receives signal -1, a loss if B receives signal 0, and a tie if B receives signal 1, and this is also a best response, establishing the claim. Clearly, (X^n, Y^n) converges to (X, Y) defined by $x_{-1} = y_{-1} = -1$, $x_0 = y_0 = 0$, and $x_1 = y_1 = 0$. But this is not an equilibrium of the Generalized Downsian Model above, because $x_{-1} = -1$ is not a best response: while $x_{-1} = -1$ produces a tie and two losses, moving to $x'_{-1} = 0$ produces two ties and one loss, increasing A's expected payoff. □

7 Mixed Strategy Characterization Results

It is difficult to say much about the nature of mixed strategy Bayesian equilibria at a general level, but in this section we take up the problems of providing some bound on the supports of equilibrium mixed strategies and of characterizing the possible atoms of mixed strategies. To this point, our results have established existence of mixed strategy Bayesian equilibria with support in M, but we have left the possibility of equilibria that put positive probability outside that interval. We next show that in the Canonical Model, even without (C4), all mixed strategy Bayesian equilibria must indeed have support in M. A consequence is that the pure strategy Bayesian equilibrium of the Probabilistic Voting Model, given by Proposition 1, is unique among all *mixed* strategy equilibria.

Theorem 10 *Under (C1)-(C3), if (G, H) is a mixed strategy Bayesian equilibrium, then it has support in M.*

Proof: Let (G, H) be a mixed strategy Bayesian equilibrium, let $\underline{x}_i = \sup\{x \in \Re : G_i(x) = 0\}$ be the lower bound of the support of G_i for each $i \in I$, and let $\underline{x} = \min_{i \in I} \underline{x}_i$ be the minimum of these lower bounds. Suppose that $\underline{x} < \underline{m}$, and take i such that $\underline{x}_i = \underline{x}$. By symmetry and interchangeability, (G, G) is also an equilibrium, so we may assume that $H = G$. Consider a sequence of pure strategies $\{X^n\}$ satisfying the following. If G_i puts positive probability on $\underline{x}$, i.e., $G_i(\underline{x}) - G_i(\underline{x})^- > 0$, then let $x_i^n = \underline{x}$ for all n. Otherwise, let $\{x_i^n\}$ be a sequence decreasing to $\underline{x}$ such that each x_i^n is in the support of G_i. Furthermore, choose x_i^n so that $\Pi_A(X^n, H|i) = \Pi_A(G, H|i)$ for all n. To see that this can be done, set x_i^0 arbitrarily and, if possible, let x_i^n be any continuity point of A's expected payoff function in the support of G_i and in the interval $[\underline{x}, \frac{\underline{x}+x_i^{n-1}}{2}]$ to satisfy the desired condition. Since there is at most a countable number of discontinuity points of A's payoff function, such a point can be found unless the support of G_i in $[\underline{x}, \frac{\underline{x}+x_i^{n-1}}{2}]$ is countable. In that case, however, any point in the support of G_i in this interval satisfies the desired condition, and there is at least one such point since $\underline{x}_i = \underline{x}$. In any case, we have $\Pi_A(X^n, H|i) = \Pi_A(G, H|i)$ for all n and $\lim_{n\to\infty} G_i(x_i^n)^- = 0$. Now consider a pure strategy X' satisfying $x_i' = \underline{m}$, and note that for n such that $x_n^i < \underline{m}$,

$$\begin{aligned}
&\Pi_A(X', H|i) - \Pi_A(X^n, H|i) \\
= \; & \sum_{j\in I} P(j|i)\Bigg[\int_{[\underline{x}, x_i^n)} \left[F_{i,j}\left(\frac{x_i^n + z}{2}\right) - F_{i,j}\left(\frac{\underline{m}+z}{2}\right)\right] H_j(dz) \\
& +(H_j(x_i^n) - H_j(x_i^n)^-)\left[\frac{1}{2} - F_{i,j}\left(\frac{x_i^n + \underline{m}}{2}\right)\right] \qquad (12)\\
& + \int_{(x_i^n, \underline{m})} \left[1 - F_{i,j}\left(\frac{\underline{m}+z}{2}\right) - F_{i,j}\left(\frac{x_i^n + z}{2}\right)\right] H_j(dz) \qquad (13)\\
& + (H_j(\underline{m}) - H_j(\underline{m})^-)\left[\frac{1}{2} - F_{i,j}\left(\frac{x_i^n + \underline{m}}{2}\right)\right] \qquad (14)\\
& + \int_{(\underline{m}, \infty)} \left[F_{i,j}\left(\frac{\underline{m}+z}{2}\right) - F_{i,j}\left(\frac{x_i^n + z}{2}\right)\right] H_j(dz)\Bigg].
\end{aligned}$$

For each $j \in I$, the first integral goes to zero, because $\lim_{n\to\infty} H_j(x_i^n)^- = \lim_{n\to\infty} G_j(x_i^n)^- = 0$. Further, the last integral is clearly non-negative. So, too, the other terms are non-negative, because $F_{i,j}\left(\frac{w+z}{2}\right) < \frac{1}{2}$ for all $w \leq \underline{m}$ and all $z < \underline{m}$. This establishes that for all $j \in I$, the expression in brackets is non-negative. It is strictly positive for $j = i$, because the bracketed terms in (12)-(14) have strictly positive limits; and the total probability mass on these terms is strictly positive since

$$\lim_{n\to\infty} H_j(\underline{m}) - H_j(x_i^n)^- \; = \; G_i(\underline{m}) - G_i(\underline{x})^- \; = \; G_i(\underline{m}) \; > \; 0.$$

Because $P(i|i) > 0$ by (C3), we conclude that

$$\Pi_A(X', H|i) \; > \; \Pi_A(X^n, H|i) \; = \; \Pi_A(G, H|i)$$

for high enough n, contradicting the assumption that (G, H) is an equilibrium. An identical argument establishes that the supports of equilibrium strategies are bounded from above by $\overline{m}$. ∎

The next example shows that the symmetry assumed in Theorem 10 is essential for the bounds on equilibrium strategies given there. Without symmetry, it is possible that one candidate, say B, essentially competes with just one type of the other candidate, A, allowing the remaining types of candidate A to win with probability one, even if they choose platforms outside the interval M.

Example 6 *Symmetry Needed for Equilibrium Bounds.* Let $I = \{-1, 1\}$, and let each $F_{i,j}$ be a uniform distribution with density 2, where priors on $I \times I$ and conditional medians are depicted below.

	$i = -1$	$i = 1$
$j = 1$	$P(-1, 1) = \epsilon$ $m_{-1,1} = -1$	$P(1, 1) = \epsilon^2$ $m_{1,1} = 1$
$j = -1$	$P(-1, -1) = 1 - \epsilon - \epsilon^2 - \epsilon^3$ $m_{-1,-1} = 0$	$P(1, -1) = \epsilon^3$ $m_{1,-1} = -1$

Note that the conditional distribution following signal pairs $(-1, 1)$ or $(1, -1)$ has support $[-1.25, -.75]$; the conditional distribution following $(-1, -1)$ has support $[-.25, .25]$; and the conditional distribution following $(1, 1)$ has support $[.75, 1.25]$. When ϵ is small, the conditional probability that candidate A receives signal $i = -1$ is close to one, regardless of B's signal. In contrast, the conditional probability that B receives signal j is close to one when A receives signal $i = j$. Let $x_{-1} = 0$, $x_1 = 1.25$, $y_{-1} = 0$, and $y_1 = -1.25$. We claim that the strategy profile (X, Y) so-defined is a Bayesian equilibrium, despite the fact that $x_1 = 1.25 > \overline{m} = 1$ and $y_1 = -1.25 < \underline{m} = -1$, violating the bound given in Theorem 10. To see this, first note that candidate A maximizes probability of winning following signal -1: moving to the left from $x_{-1} = 0$ only decreases A's probability of winning in case B receives signal -1; and moving far enough to increase A's probability of winning in case B receives signal 1 means A must position at $x'_{-1} < -.75$, but then A would win with probability zero in case B receives the more likely signal -1. Similarly, A maximizes probability of winning following signal 1: A already wins with probability one in case B receives signal 1; and A cannot increase the probability of winning in case B receives signal -1 without moving to the left of $y_{-1} = 0$, but then A would win with probability zero in case B receives the more likely signal 1. A symmetric argument for B establishes the claim. $\square$

Theorem 10 applies to the Probabilistic Voting Model, and more generally to the Common Knowledge Model with symmetry conditions (C1)-(C3) added. Moreover, decomposing the

Probabilistic Voting Model into its component games, it applies to each one separately. Since $P(i,j) > 0$ implies $i = j$, and since the set of medians for the component game corresponding to the signal pair (i,i) is just the singleton consisting of m_i, we conclude that the unique mixed strategy Bayesian equilibrium of the component game is the point mass on m_i for both candidates. In other words, the unique pure strategy Bayesian equilibrium characterized in Proposition 1 is also unique among all mixed strategy Bayesian equilibria. This corollary, with Theorem 7, gives us a strong continuity result, paralleling Theorem 9, for the Probabilistic Voting Model: In models close to the Probabilistic Voting Model, mixed strategy equilibria must be close, in the sense of weak convergence, to the pure strategy equilibrium in which the candidates locate at m_i following signal i.

Corollary 2 *In the Probabilistic Voting Model, if (G,H) is a mixed strategy Bayesian equilibrium, then the candidates locate at m_i with probability one following signal $i \in I$, i.e., $G_i(m_i) = G_i(m_i)^- = H_i(m_i) - H_i(m_i)^- = 1$ for all $i \in I$.*

Bernhardt, Duggan, and Squintani (2003) show that, in the Stacked-Uniform Model, mixed strategy equilibrium distributions may admit atoms following some signal realizations (while the distributions following other realization may be continuous). In that model, those atoms occur only at the conditional medians, $m_{i,i}$. Our next result shows that this observation generalizes, giving weak conditions for the Canonical Model that pin down atoms to the conditional medians. In addition to (C1)-(C4), we impose the following "monotone likelihood ratio" condition on conditional signal probabilities.

(C8) For all signals $i, j, i', j' \in I$, if $i \prec i'$ and $j \prec j'$, then

$$P(j|i')P(j'|i) \quad \leq \quad P(j|i)P(j'|i').$$

As long as $P(j|i)$ and $P(j|i')$ are positive, we can rewrite condition (C8) more intuitively as

$$\frac{P(j'|i)}{P(j|i)} \leq \frac{P(j'|i')}{P(j|i')},$$

which gives the monotone likelihood condition its name.

A reasonable interpretation of the signals in our model is that they indicate the ideological leanings of the electorate, i.e., whether μ is likely to be located more to the left or more to the right. Under that interpretation, the following stochastic dominance condition, which presumes (C1)-(C4), is natural.

(C9) For all signals $i, i' \in I$ with $i \prec i'$, for all $j \in I$, and for all $z \in M$ with $0 < F_{i',j}(z) < 1$, we have $F_{i',j}(z) < F_{i,j}(z)$.

By continuity, this implies that for signals $i \prec i'$, $F_{i',j}(z) \leq F_{i,j}(z)$, for all $z \in M$. Condition (C9) is implied by (C4) in the Stacked-Uniform Model when $a \geq b_N - b_1$: In that case, $i \prec i'$ implies $m_{i',j} > m_{i,j}$, so that

$$F_{i',j}(z) \;=\; \frac{1}{2} + \frac{z - m_{i',j}}{2a} \;<\; \frac{1}{2} + \frac{z - m_{i,j}}{2a} \;=\; F_{i,j}(z)$$

which yields the condition.

Lemma 4, which we prove in the appendix, establishes a key consequence of (C8) and (C9).

Lemma 4 *In the Canonical Model, assume that conditions (C8) and (C9) hold. For each $j \in I$, let $\alpha_j \in [0, 1]$. Then, for all $i, i' \in I$ with $i \prec i'$ and for all $z \in M$ with*

$$\alpha_j P(j|i)P(j|i') \;>\; 0 \quad \textit{and} \quad F_{i',j}(z) \;<\; F_{i,j}(z)$$

for at least one j, we have

$$\frac{\sum_{j\in I} \alpha_j P(j|i)F_{i,j}(z)}{\sum_{j\in I} \alpha_j P(j|i)} \;>\; \frac{\sum_{j\in I} \alpha_j P(j|i')F_{i',j}(z)}{\sum_{j\in I} \alpha_j P(j|i')}.$$

Note that Lemma 4 reinforces (C4): Setting $\alpha_j = 1$ for all $j \in K$ and $\alpha_j = 0$ otherwise, we see that $F_{i',K}$ stochastically dominates $F_{i,K}$. Lemma 4 allows us to prove (in the appendix) a final lemma on the location of mass points of equilibrium mixed strategies. It parallels, under the extra conditions of (C8) and (C9), Lemma 3.

Lemma 5 *In the Canonical Model, assume that conditions (C8) and (C9) hold. Let (G, H) be a mixed strategy Bayesian equilibrium. For all $z \in M$, if both candidates place positive probability mass on z, that is, $G_i(z) - G_i(z)^- > 0$ for some $i \in I$ and $H_j(z) - H_j(z)^- > 0$ for some $j \in I$ with $P(i, j) > 0$, then $z = m_{i,j}$.*

The intuition behind this lemma is simple. Suppose that candidates A and B both put positive mass on the same point $z \in M$ following signal realizations, i and j. Lemma 4 allows us to assume that i and j are the only signal realizations after which the candidates put positive mass on z. The argument then proceeds as in Lemma 3. Conditional on signals i and j, each candidate expects to choose z with positive probability, and if z is not equal to $m_{i,j}$, then a candidate, say A, can transfer probability mass from z and move it toward $m_{i,j}$ by an arbitrarily small amount. This increases A's expected payoff discretely when B chooses z, and it affects A's expected payoff continuously otherwise. Therefore, a small enough deviation increases A's expected payoff.

We now derive our restriction on atoms of mixed strategy Bayesian equilibria: In the Canonical Model with (C8) and (C9), the only possible atom of an equilibrium distribution, G_i or H_i, is the conditional median $m_{i,i}$. Save for the added assumptions of (C8) and (C9), this result generalizes Theorem 1.

Theorem 11 *In the Canonical Model, assume that conditions (C8) and (C9) hold. Let* (G, H) *be a mixed strategy Bayesian equilibrium. If* $G_i(z) - G_i(z)^- > 0$ *for some* $i \in I$*, then* $z = m_{i,i}$*. If* $H_j(z) - H_j(z)^- > 0$ *for some* $j \in I$*, then* $z = m_{j,j}$*.*

Proof: Let (G, H) be a mixed strategy Bayesian equilibrium, and suppose $G_i(z) - G_i(z)^- > 0$ for some $i \in I$, but $z \neq m_{i,i}$. By symmetry and interchangeability, (G, G) is an equilibrium, and (C3) imposes $P(i, i) > 0$. By Theorem 10, we must have $z \in M$. But then Lemma 5 implies $z = m_{i,i}$, a contradiction. ∎

Theorem 11 does not quite allow us to use differentiable methods to analyze mixed strategy Bayesian equilibria. While the result limits the potential discontinuities of equilibrium mixed strategies to a finite set, there may be other points at which an equilibrium distribution H_j is non-differentiable, albeit continuous. The Cantor-Lebesgue function (see Wheeden and Zygmund, 1977) is an example of a continuous distribution that puts probability one on its points of non-differentiability, so this technical problem is potentially significant. In applications, it may be helpful to restrict attention to a subset of mixed strategy equilibria: We say a mixed strategy pair (G, H) is *regular* if for all $i \in I$ and all $z \in \Re$, either G_i is differentiable at z or it is discontinuous at z, and similarly for H_i. This restriction eliminates the problem anticipated above, at the cost of possibly omitting pathological equilibria.

Suppose (G, H) is a regular mixed strategy Bayesian equilibrium of the Canonical Model. Under the conditions of Theorem 11, we can decompose the probability measure generated by H_j into a degenerate measure with mass $H_j(m_{j,j}) - H_j(m_{j,j})^-$ on $m_{j,j}$ and an absolutely continuous measure with density h_j defined at all but at most a finite number of points.

Candidate A's expected payoff from pure strategy X against H, conditional on signal i, is

$$\begin{aligned}
\Pi_A(X,H|i) \;=\; & \sum_{j\in I: m_{j,j}<x_i} P(j|i)\left[1-F_{i,j}\left(\frac{x_i+m_{j,j}}{2}\right)\right](H_j(m_{j,j})-H_j(m_{j,j})^-) \\
& + \sum_{j\in I: m_{j,j}=x_i} \frac{1}{2}P(j|i)(H_j(m_{j,j})-H_j(m_{j,j})^-) \\
& + \sum_{j\in I: x_i<m_{j,j}} P(j|i)F_{i,j}\left(\frac{x_i+m_{j,j}}{2}\right)(H_j(m_{j,j})-H_j(m_{j,j})^-) \\
& + \sum_{j\in I} P(j|i)\left[\int_{-\infty}^{x_i}\left[1-F_{i,j}\left(\frac{x_i+z}{2}\right)\right]h_j(z)\,dz+\int_{x_i}^{\infty}F_{i,j}\left(\frac{x_i+z}{2}\right)h_j(z)\,dz\right].
\end{aligned}$$

Note that this expected payoff is differentiable at x_i if there is no j such that $x_i = m_{j,j}$. Indeed, it is enough that if $x_i = m_{j,j}$, then H_j does not put positive mass on $m_{j,j}$.

Under these conditions, the usual first order condition must be satisfied at any platform x_i in the support of candidate A's equilibrium distribution following signal i:

$$\begin{aligned}
0 \;=\; & \sum_{j\in I: m_{j,j}<x_i} \frac{1}{2}P(j|i)\left[-f_{i,j}\left(\frac{x_i+m_{j,j}}{2}\right)(H_j(m_{j,j})-H_j(m_{j,j})^-)\right] \\
& + \sum_{j\in I: x_i<m_{j,j}} \frac{1}{2}P(j|i)\left[f_{i,j}\left(\frac{x_i+m_{j,j}}{2}\right)(H_j(m_{j,j})-H_j(m_{j,j})^-)\right] \\
& + \sum_{j\in I} P(j|i)\left[\int_{-\infty}^{x_i} -\frac{1}{2}f_{i,j}\left(\frac{x_i+z}{2}\right)h_j(z)\,dz+(1-F_{i,j}(x_i))h_j(x_i)\right. \\
& \left. + \int_{x_i}^{\infty}\frac{1}{2}f_{i,j}\left(\frac{x_i+z}{2}\right)h_j(z)\,dz - F_{i,j}(x_i)h_j(x_i)\right].
\end{aligned}$$

Bernhardt, Duggan, and Squintani (2003) use these observations to construct a regular mixed strategy equilibrium in the Stacked-Uniform Model and to derive its properties.

8 Conclusion

This paper provides theoretical results on elections with privately informed candidates, where the source of private information may be a candidate's personal experiences or polls conducted by the candidate's campaign organization or political party. We characterize the unique pure strategy Bayesian equilibrium of the electoral game, when it exists, and we give results on the existence and continuity properties of mixed strategy Bayesian equilibria, as well as bounds on the supports of mixed strategy equilibria and restrictions on equilibrium atoms. By setting the analysis within a general framework, not only do we strengthen the foundation of these results, but we also open the possibility of developing special cases of interest in applications. We do just that for the Downsian Model and the Probabilistic Voting Model, perhaps the most common electoral models used in applications. In particular, we illustrate a type of fragility of the Downsian equilibrium, and provide robustness results in mixed strategies for both models: if a small amount of private information is added to either model, there exist mixed strategy Bayesian equilibria, and in all equilibria a candidate must place probability near one close to the median conditional on his signal.

We consider other special cases, the Stacked-Uniform Model and the Shape-Invariant Model, where added structure promises further insights into the workings of elections with private information. Bernhardt, Duggan, and Squintani (2003) develop the Stacked-Uniform Model, given an explicit solution for the (essentially unique) mixed strategy Bayesian equilibrium in this model: They show that, when the pure strategy equilibrium fails to exist, a candidate receiving a "central" signal i locates at the median conditional on both candidates receiving that signal, $m_{i,i}$, while candidates who receive "extreme" signals mix, moderating their platforms relative to the pure strategy equilibrium choices. The tractability of the Stacked-Uniform Model also permits a number of comparative statics and an analysis of voter welfare. It is shown, for example, that while candidate locate more extremely than they would if they simply targeted the median voter on the basis of their private information, all voters would be better off if the candidates chose *more* extreme platforms: Assuming voter preferences are correlated with the median voter's, they all prefer to give the median voter a degree of choice when the candidates receive different signals, and the equilibrium dispersion between the candidates' platforms in that case is suboptimal.

The model admits a variety of other types of structure that may be of interest to develop. A more detailed description of the polling process or of the determinants of the median voter's ideal point, for example, would open the possibility of further comparative statics results. Further, several extensions of the model are of potential importance. First, because the strategic value of better information is always positive for candidates, it is straightforward to endogenize the choice of costly polling technologies by candidates. Second, it would be worthwhile to determine how outcomes are affected when candidates have ideological preferences, and to endogenize contributions by ideologically-motivated lobbies to fund polling by candidates. Finally, as Ledyard (1989) observes, it would be useful to uncover how equilibrium outcomes are affected when candidates choose platforms sequentially, so the second candidate can see where the first locates, and hence can unravel the latter's signal, before locating.

A Appendix

Lemma 1 *Let (G, H) be any pair of mixed strategies. For all $s \in S$ and all $w, z \in \Re$, one of three possibilities obtains: Either*

$$\begin{aligned}&\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][F_{s,t}(w)] \\ &\quad = \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-] = \sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][1 - F_{s,t}(w)],\end{aligned}$$

or

$$\begin{aligned}&\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][F_{s,t}(w)] \\ &\quad < \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-] < \sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][1 - F_{s,t}(w)],\end{aligned}$$

or the reverse inequalities hold. Likewise for G and all $t \in T$ and all $w, z \in \Re$.

Proof: Given $s \in S$ and $z \in \Re$, the first possibility clearly obtains if $P(t|s)[H_t(z)-H_t(z)^-] = 0$ for all $t \in T$. Suppose $P(t|s)[H_t(z) - H_t(z)^-] > 0$ for some $t \in T$, and define the function F^* as follows:

$$F_s^*(w) = \frac{\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-]F_{s,t}(w)}{\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-]}.$$

Since each $F_{s,t}$ is continuous at w, the three possibilities above correspond to the three possibilities $F_s^*(w) = 1/2$, $F_s^*(w) < 1/2$, and $F_s^*(w) > 1/2$. ∎

Lemma 2 *Let (G, H) be a mixed strategy Bayesian equilibrium. For all $z \in \Re$, if $G_{s'}(z) - G_{s'}(z)^- > 0$ for some $s' \in S$ and $H_{t'}(z) - H_{t'}(z)^- > 0$ for some $t' \in T$ with $P(s', t') > 0$, then*

$$\begin{aligned}&\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][F_{s',t}(z)] \\ &\quad = \frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-] = \sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][1 - F_{s',t}(z)].\end{aligned}$$

and

$$\begin{aligned}&\sum_{s\in S} P(s|t')[G_s(z) - G_s(z)^-][F_{t',s}(z)] \\ &\quad = \frac{1}{2}\sum_{s\in S} P(s|t')[G_s(z) - G_s(z)^-] = \sum_{s\in S} P(s|t')[G_s(z) - G_s(z)^-][1 - F_{t',s}(z)].\end{aligned}$$

Proof: We prove the first equalities. If they do not hold for some z and some s' and t' with $P(s',t') > 0$, then, by Lemma 1, we may assume that

$$\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][1 - F_{s',t}(z)] \quad > \quad \frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-] \tag{15}$$

or

$$\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][F_{s',t}(z)] \quad > \quad \frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-].$$

We focus on the first inequality, as a symmetric proof addresses the second. For each $t \in T$, let λ_t denote the probability measure generated by the distribution H_t, let μ_t denote the degenerate measure with mass $H_t(z) - H_t(z)^-$ on z, and let $\nu_t = \lambda_t - \mu_t$. Let $\{x^n\}$ be a sequence decreasing to z, and let G^n be the mixed strategy defined by replacing G_s in G with the point mass on x^n. Let

$$\pi_t(w) \quad = \quad \pi_A(z, w|s', t)$$

denote A's probability of winning using z when B receives signal t and chooses platform w, and let

$$\pi_t^n(w) \quad = \quad \pi_A(x^n, w|s', t)$$

denote A's probability of winning using x^n when B receives signal t and chooses platform w. Then

$$\begin{aligned}\Pi_A(G^n, H|s') - \Pi_A(G, H|s') \quad &= \quad \sum_{t\in T} P(t|s') \int [\pi_t^n(w) - \pi_t(w)]\,\lambda_t(dw) \\ &= \quad \sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-]\left[1 - F_{s',t}\left(\frac{z + x^n}{2}\right) - \frac{1}{2}\right] \\ &\quad\quad + \sum_{t\in T} P(t|s') \int [\pi_t^n(w) - \pi_t(w)]\,\nu_t(dw).\end{aligned}$$

Since $\pi_t^n - \pi_t \to 0$ almost everywhere (ν_t), the corresponding integral terms above converge to zero. Thus,

$$\begin{aligned}&\lim_{n\to\infty} \Pi_A(G^n, H|s') - \Pi_A(G, H|s') \\ &= \quad \sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][1 - F_{s',t}(z)] - \frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-] \\ &> \quad 0,\end{aligned}$$

by (15). It follows that $\Pi_A(G^n, H|s) > \Pi_A(G, H|s)$ for high enough n, a contradiction. ∎

Lemma 3 *In the Canonical Model, let* (X, Y) *be a pure strategy Bayesian equilibrium. If* $x_i = y_j$ *for some* $i, j \in I$ *with* $P(i, j) > 0$*, then* $x_i = y_j = m_{i,j}$*.*

Proof: Let G be a mixed strategy for A such that, for all $i \in I$, $G_i(x_i) - G_i(x_i)^- = 1$, and define H for B analogously. Let $I' = \{i \in I : x_i = z\}$, and let $J' = \{j \in I : y_j = z\}$. Take any $i \in I'$ and any $j \in J'$ with $P(i, j) > 0$. Lemma 2 implies

$$\sum_{j' \in J'} \frac{P(j'|i)}{P(J'|i)}[F_{i,j'}(z)] \quad = \quad \frac{1}{2}, \tag{16}$$

where we use $H_{j'}(z) - H_{j'}(z)^- = 1$ for $j' \in J'$ and $H_{j'}(z) - H_{j'}(z)^- = 0$ otherwise. Thus, $z = m_{i,J'}$. If there exists $i' \in I'$ with $i' \neq i$ and $P(i', j) > 0$, then (16) must hold for i' as well, implying $z = m_{i',J'}$. Since $P(J'|i) > 0$ and $P(J'|i') > 0$, (C4) implies that i and i' have the same conditional distributions. An analogous argument for candidate B establishes that $j' \in J'$ and $P(i, j') > 0$ imply that j and j' have the same conditional distributions. Now take any $j' \in J'$ such that $P(j'|i) > 0$. This implies $P(i, j') > 0$, so $F_{i,j'} = F_{i,j}$. Therefore, (16) reduces to $F_{i,j}(z) = 1/2$, i.e., $z = m_{i,j}$. ∎

Lemma 4 *In the Canonical Model, assume (C8) and (C9). For each* $j \in I$*, let* $\alpha_j \in [0, 1]$*. For all* $i, i' \in I$ *with* $i \prec i'$ *and for all* $z \in M$ *with*

$$\alpha_j P(j|i)P(j|i')F_{i',j}(z) \ > \ 0 \quad \textit{and } F_{i',j}(z) < F_{i,j}(z)$$

for at least one j*, we have*

$$\frac{\sum_{j \in I} \alpha_j P(j|i)F_{i,j}(z)}{\sum_{j \in I} \alpha_j P(j|i)} \quad > \quad \frac{\sum_{j \in I} \alpha_j P(j|i')F_{i',j}(z)}{\sum_{j \in I} \alpha_j P(j|i')}.$$

Proof: Take $i, i' \in I$, $z \in \Re$, and α_j's as in the statement of the lemma. By assumption, $\alpha_j P(j|i) > 0$ and $\alpha_j P(j|i') > 0$ for some j, so cross multiply and rewrite the inequality as

$$\sum_{j,j' \in I} \alpha_j \alpha_{j'} P(j|i)P(j'|i')F_{i,j}(z) > \sum_{j,j' \in I} \alpha_j \alpha_{j'} P(j|i')P(j'|i)F_{i',j}(z).$$

We compare the two sides of the inequality one pair $\{j, j'\}$ at a time. For $j = j'$, we have

$$\alpha_j^2 P(j|i)P(j|i')F_{i,j}(z) \quad \geq \quad \alpha_j^2 P(j|i)P(j|i')F_{i',j}(z)$$

from (C9). Moreover, there is at least one j such that $\alpha_j^2 P(j|i)P(j|i) > 0$ and $F_{i,j}(z) > F_{i',j}(z)$, which gives us a strict inequality. For distinct j and j', say $j < j'$, we want to show that

$$\begin{aligned}&\alpha_j\alpha_{j'}[P(j|i)P(j'|i')F_{i,j}(z) + P(j'|i)P(j|i')F_{i,j'}(z)]\\ \geq\;\; &\alpha_j\alpha_{j'}[P(j|i')P(j'|i)F_{i',j}(z) + P(j'|i')P(j|i)F_{i',j'}(z)].\end{aligned}$$

Note that by (C9), we have

$$F_{i,j}(z) \;\geq\; \max\{F_{i,j'}(z), F_{i',j}(z)\} \;\geq\; \min\{F_{i,j'}(z), F_{i',j}(z)\} \;\geq\; F_{i',j'}(z),$$

and therefore

$$F_{i,j}(z) - F_{i',j'}(z) \quad\geq\quad F_{i',j}(z) - F_{i,j'}(z).$$

Then (C8) implies

$$P(j|i)P(j'|i')(F_{i,j}(z) - F_{i',j'}(z)) \quad\geq\quad P(j|i')P(j'|i)(F_{i',j}(z) - F_{i,j'}(z)),$$

which yields the desired inequality. ∎

Lemma 5 *In the Canonical Model, assume (C8) and (C9). Let (G, H) be a mixed strategy Bayesian equilibrium. For all $z \in M$, if $G_i(z) - G_i(z)^- > 0$ for some $i \in I$ and $H_j(z) - H_j(z)^- > 0$ for some $j \in I$ with $P(i,j) > 0$, then $z = m_{i,j}$.*

Proof: Let (G, H) be a mixed strategy Bayesian equilibrium, and take any $z \in M$. Define the sets

$$\begin{aligned}I' &= \{i \in I : G_i(z) - G_i(z)^- > 0\}\\ J' &= \{j \in I : H_j(z) - H_j(z)^- > 0\}.\end{aligned}$$

Take any $i \in I'$ and $j \in J'$ such that $P(i,j) > 0$. Lemma 2 implies

$$\sum_{j'\in I} P(j'|i)[H_{j'}(z) - H_{j'}(z)^-][F_{i,j'}(z)] \;=\; \frac{1}{2}\sum_{j'\in I} P(j'|i)[H_{j'}(z) - H_{j'}(z)^-]. \qquad (17)$$

Setting $\alpha_{j'} = H_{j'}(z) - H_{j'}(z)^-$, this is equivalent to

$$\frac{\sum_{j'\in I}\alpha_{j'}P(j'|i)F_{i,j'}(z)}{\sum_{j'\in I}\alpha_{j'}P(j'|i)} \;=\; \frac{1}{2}.$$

Note that (17) must hold for j as well from candidate B's perspective. If there exists $\hat{j} \in J'$ such that $P(\hat{j}|i) > 0$, then (17) also holds for $\hat{j}$. Letting $\beta_{i'} = G_{i'}(z) - G_{i'}(z)^-$, we then have

$$\frac{\sum_{i' \in I} \beta_{i'} P(i'|j) F_{j,i'}(z)}{\sum_{i' \in I} \beta_{i'} P(i'|j)} = \frac{\sum_{i' \in I} \beta_{i'} P(i'|\hat{j}) F_{\hat{j},i'}(z)}{\sum_{i' \in I} \beta_{i'} P(i'|\hat{j})}. \tag{18}$$

We claim that $F_{i,j}(z) = F_{i,\hat{j}}(z)$. Otherwise, we have $\beta_i P(i|j) P(i|\hat{j}) > 0$ and $F_{i,j}(z) \neq F_{i,\hat{j}}(z)$. Then (C4) implies either $j \prec \hat{j}$ or $\hat{j} \prec j$, and Lemma 4 contradicts (18). Therefore, for all $\hat{j} \in J'$ with $P(\hat{j}|i) > 0$, we have $F_{i,\hat{j}}(z) = F_{i,j}(z)$, and (17) reduces to $F_{i,j}(z) = 1/2$, i.e., $z = m_{i,j}$. ∎

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