Core Convergence in Dynamic Legislative Bargaining

John Duggan   Tasos Kalandrakis

November 18, 2008

Outline

Background

The positive theory of social choice in political science: Riker, Ordeshook, Hinich, Plott, McKelvey, Schofield.

The enterprize: make predictions on the basis of majority voting. Related to tournaments, but more structure. (Typically, $X \subseteq \Re^d$, plus voters.)

The conclusion: core is empty, top cycle is indiscriminate.

Background (cont.)

Alternative responses: (1) refinements of top cycle (looking for "stable" sets), or (2) stochastic solutions (focusing on dynamics), or (3) non-cooperative game theory.

Ferejohn, McKelvey, and Packel (1984) take the dynamic approach to social choice theory, but their voters are myopic, and they model the agenda as a draw from an exogenous family of distributions.

We consider a game-theoretic model of dynamic bargaining with farsighted agents and endogenous proposals.

Spatial Model

Elements:

X	alternatives (subset of $\Re^d$)
N	voters $i = 1, \ldots, n$ (n odd)
u_i	utility function for i (with unique ideal point $\hat{x}^i$)
P	strict majority preference, i.e., xPy if and only if $u_i(x) > u_i(y)$ for $> \frac{n}{2}$ voters

We say preferences are *Euclidean* if $u_i(x) = -||\hat{x}^i - x||^2$.

Spatial Model (cont.)

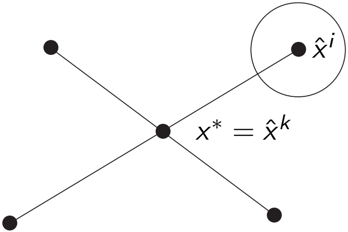
Solution sets:

$K(P)$	the *core*, i.e., alternatives x such that for all $y \neq x$, xPy
$TC(P)$	the *top cycle*, i.e., x such that for all $y \neq x$, there exist $x_1, \ldots, x_{k-1}$ such that $xPx_1Px_2 \ldots x_{k-1}Py$

Spatial Model (cont.)

Theorem (Plott)

Assume Euclidean preferences with distinct ideal points. If the core is nonempty, then it consists of a single alternative, say x^*. Further, x^* is the ideal point of some voter, say k, and radial symmetry is satisfied at x^*.



Spatial Model (cont.)

Theorem (McKelvey)

Assume $X = \Re^d$ and Euclidean preferences. If $K(P) = \emptyset$, then $TC(P) = \Re^d$.

Thus, even when the core is almost nonempty, starting from any y, we may end up at any x via a finite sequence $xPx_1Px_2 \cdots x_{k-1}Py$.

Issues: (1) core is almost always empty, and (2) top cycle is not useful in that case.

Markov Chains

X	compact subset of $\Re^d$
$P(x, Y)$	probability of moving to state in Y from x ($P(x, \cdot)$ a probability measure, $P(\cdot, Y)$ a measurable function)
$T^*\mu(Y)$	probability tomorrow's state lies in Y given distribution μ today, i.e.,

$$T^*\mu(Y) = \int_X P(x, Y)\mu(dx)$$

$T^{*m}\mu(Y)$	probability m periods hence.

Markov Chains (cont.)

A probability measure μ is *invariant* if $\mu = T^*\mu$,

Assume *Doeblin's condition* with respect to Lebesgue measure: There is an iteration m and $\epsilon > 0$ such that for all Y,

$$\lambda(Y) \leq \epsilon \quad \text{implies} \quad \begin{array}{c} P^m(x, Y) \leq 1 - \epsilon \\ \text{for all } x. \end{array}$$

A set E is *ergodic* if (1) for all $x \in E$, $P(x, E) = 1$, and (2) this is not true of any $E' \subseteq E$ with $\lambda(E') < \lambda(E)$.

Then there is a finite number of ergodic sets E_j, and each ergodic set is associated with a unique invariant distribution μ_j.

Markov Chains (cont.)

Given any initial μ, the sequence of long run averages converges in total variation to an invariant distribution μ^*, i.e.,

$$\left| \mu^*(Y) - \frac{1}{t} \sum_{m=1}^{t} T^{*m} \mu(Y) \right| \to 0,$$

uniformly in Y.

Say P is *aperiodic* if there are not pairwise disjoint, nonempty sets $C_1, \ldots, C_k$ with $k \geq 2$ such that for all $x \in C_j$, $P(x, C_{j+1}) = 1$.

In this case, convergence is stronger: there exist constants $\rho < 1$ and c such that for all Y and all t,

$$|\mu^*(Y) - T^{*t} \mu(Y)| \quad \leq \quad c \rho^t.$$

Dynamic Social Choice

In the spatial model, let $P^*(x, \cdot)$ be the uniform distribution on the set of y such that yPx.

Theorem (Ferejohn, McKelvey, Packel)

Assume X is compact and preferences are Euclidean. Then P^* satisfies Doeblin's condition and is aperiodic. Further, the ergodic set is unique. If the core is nonempty, then it is $K(P)$, and otherwise, it is X. Thus, there is a unique invariant distribution μ^*, and for every initial distribution μ, the sequence $\{T^{*t}\mu\}$ converges to μ^* geometrically in total variation.

Dynamic Social Choice (cont.)

Let $\xi = (\hat{x}^i)_{i \in N}$ denote a profile of ideal points.

Say $\{\xi^m\}$ is *approximately canonical* if it converges to ξ satisfying radial symmetry around some x^*.

Theorem (Ferejohn, McKelvey, Packel)

Assume X is compact and preferences are Euclidean, let $\{\xi^m\}$ be approximately canonical, and let μ^m be the unique invariant distribution for ξ^m. Then $\{\mu^m\}$ converges weak to the unit mass on x^*.*

Dynamic Social Choice (cont.)

When radial symmetry is disturbed, the core is empty. But the long run distribution of outcomes is uniquely defined and will be concentrated near the location of the initial core alternative.

Issues: (1) voters are myopic, and (2) distribution of alternatives on the agenda is exogenous.

Outline

Model

- Status quo q in $\Re^d$ and preference shocks $\theta = (\theta_1, \ldots, \theta_n)$ are publicly observed.
- Each individual i is drawn with probability p_i to make a propose an alternative in $X \subseteq \mathbb{R}^d$.
- Selected individual proposes an alternative $y \in X \cup \{q\}$.
- Individuals vote using majority rule to accept y or reject in favor of q.

The model (cont.)

⋮

The current period outcome is

$$x = \begin{cases} y & \text{if } |\{j \mid j \text{ accepts }\}| > \frac{n}{2} \\ q & \text{else.} \end{cases}$$

Individual i receives utility $\delta_i^{t-1}(u_i(x) + \theta_i \cdot x)$.

New status quo q' is drawn from density $g(\cdot|x)$, and new preference parameters $\theta' = (\theta'_1, \ldots, \theta'_n)$ are drawn from density $f(\cdot)$.

The above procedure is repeated in the next period.

▸ Assumptions

Stationary strategies

Proposal strategies: $\pi_i(q, \theta) \in X \cup \{q\}$

Voting strategies: $\alpha_i(y, q, \theta) \in \{a, r\} \ldots$

or in terms of an acceptance set:

$$A_i(q, \theta) \quad = \quad \{y \in X \cup \{q\} \mid \alpha_i(y, q, \theta) = a\}$$

Social acceptance set:

$$A(q, \theta) \quad = \quad \bigcup_{C \in \mathscr{D}} \bigcap_{i \in C} A_i(q, \theta)$$

Continuation values

Dynamic payoff from outcome x and shock θ_i is:

$$U_i(x, \theta_i) \;\; = \;\; (1 - \delta_i)(u_i(x) + \theta_i \cdot x) + \delta_i v_i(x),$$

where $v_i(x)$ is expected payoff beginning tomorrow given outcome x today.

Strategies satisfy *no delay* if for all i, all q, and all θ, $\pi_i(q, \theta) \in A(q, \theta)$.

Then continuation values satisfy

$$v_i(x) \;\; = \;\; \int_q \int_\theta \sum_j p_j U_i(\pi_j(q, \theta), \theta_i) f(\theta) g(q|x) d\theta dq.$$

Stationary legislative equilibrium

Optimal voting strategies: for all q, all θ, and all i,

$$A_i(q,\theta) \quad = \quad \{y \in X \cup \{q\} \mid U_i(y,\theta_i) \geq U_i(q,\theta)\}.$$

Optimal proposal strategies: for all q, all θ, and all i, $\pi_i(q,\theta)$ solves

$$\max_{y \in X \cup \{q\}} U_i(y,\theta_i)$$
$$\text{s.t. } y \in A(q,\theta).$$

This adds several refinements to stationary Markov perfect equilibrium: 1) pure strategies, 2) stage undominated voting, 3) voter deference, and 4) no delay.

Existence

Theorem

There exists a stationary legislative equilibrium such that:

1. *continuation values are smooth.*
2. *proposals are almost always strictly best.*
3. *proposal strategies are differentiable almost everywhere.*

Comments on Existence

Long run equilibrium policies

Given stationary strategies, the transition probability

$$P(x, Y) = \int_q \int_\theta \sum_i p_i I_Y(\pi_i(q, \theta)) f(\theta) g(q|x) d\theta dq$$

summarizes the law of motion of outcomes over time.

The distribution of outcomes next period, given distribution μ today, is given by

$$T^* \mu(Y) = \int_x P(x, Y) \mu(dx).$$

Long run equilibrium policies (cont.)

Theorem

1. *The transition probability P satisfies Doeblin's condition.*
2. *If $g(x|x) > 0$ for every alternative x, then P is aperiodic.*
3. *If for every pair of alternatives x, y, there exists q such that $g(q|x)g(q|y) > 0$, then P admits a unique invariant distribution.*

Outline

Set up

A model is $\gamma = ((p_i, u_i, \delta_i)_{i \in N}, X, f, g)$.

Let $\mathcal{X}$ be compact, and let $\Gamma_{\mathcal{X}}$ denote the set of models γ such that $X \subseteq \mathcal{X}$.

Let $E(\gamma)$ be the set of stationary legislative equilibria in model γ.

A model γ is ϵ-*canonical* if there exist $(k, (\hat{x}^i)_{i \in N}, \delta)$ such that

1. $(\hat{x}^i)_{i \in N}$ satisfies radial symmetry through $\hat{x}^k$,
2. for all $x \in \mathcal{X}$, $|u_i(x) + ||\hat{x}^i - x||^2| < \epsilon$,
3. $\frac{|\delta_i - \delta|}{1 - \delta_i} < \epsilon$,
4. $\text{supp} f \subseteq B_\epsilon(0)$,
5. $\text{supp} g(\cdot|x) \subseteq B_\epsilon(x)$.

Lower Bound on Payoffs

Lemma

For all $\lambda > 0$, there exists $\tilde{\epsilon} > 0$ such that for all $\epsilon < \tilde{\epsilon}$, all ϵ-canonical models $\gamma \in \Gamma_{\mathcal{X}}$, all stationary strategy profiles σ, all θ with $f(\theta) > 0$, and all $y, z \in \mathcal{X}$,

$$U_k(y, \theta_k) > U_k(z, \theta_k) + \lambda \quad \Rightarrow \quad |\{i : U_i(y, \theta_i) > U_i(z, \theta_i)\}| > \frac{n}{2}$$

and

$$|\{i : U_i(y, \theta_i) \geq U_i(z, \theta_i)\}| > \frac{n}{2} \quad \Rightarrow \quad U_k(y, \theta_k) > U_k(z, \theta_k) - \lambda$$

Lower Bound on Payoffs (cont.)

In equilibrium, we then have: for all y, all q, and all θ, legislator k's payoff when selected to propose is at least $U_k(y, \theta_k) - \lambda$.

Likewise, legislator k's payoff is at least $U_k(q, \theta_k) - \lambda$ when another legislator is selected and the status quo is q.

Theorem

For all $\lambda > 0$, there exists $\overline{\epsilon} > 0$ such that for all $\epsilon < \overline{\epsilon}$, all ϵ-canonical models $\gamma \in \Gamma_{\mathcal{X}}$, all stationary legislative equilibria $\sigma \in E(\gamma)$, and all $y \in \mathcal{X}$,

$$v_k(y) \quad \geq \quad u_k(y) - \frac{\lambda}{1 - \delta_i}.$$

Core Convergence

A sequence $\{\gamma^m\}$ in $\Gamma_{\mathcal{X}}$ is *approximately canonical* if there is a sequence $\{\epsilon^m\}$ such that

1. for all m, γ^m is ϵ^m-canonical with respect to fixed $(k, (\hat{x}^i)_{i \in N}, \delta)$ with $\delta < 1$,
2. for all m, $g^m(x|x) > 0$,
3. $p^{k,m} \to p^k > 0$,
4. $\min\{||x - \hat{x}^k|| : x \in X^m\} \to 0$,
5. $\epsilon^m \to 0$.

Core Convergence (cont.)

Theorem

Let $\{\gamma^m\}$ be approximately canonical, and for each m, let $\sigma^m \in E(\gamma^m)$, and let μ^m be an invariant distribution corresponding to σ^m. Then $\{\mu^m\}$ converges weak to the unit mass on $\hat{x}^k$.*

Consider any sequence $\{\mu^m\}$ of equilibrium invariant distributions and any $\eta > 0$. It suffices to show that $\mu^m(B_\eta(\hat{x}^k)) \to 1$.

Let P_m^t be the t-step transition for the equilibrium in γ^m. We can find a sequence $\{y_m\}$ such that

$$\lim_m P^t(y_m, B_\eta(\hat{x}^k)) = 1 \quad \text{and} \quad \lim_t P^t(y_m, B_\eta(\hat{x}^k)) = \mu^m(B_\eta(\hat{x}^k)).$$

Core Convergence (cont.)

$$
m \left\{ \overbrace{\begin{array}{ccc}
P_1^1(y_1, B_\eta(\hat{x}^k)) & \cdots & P_1^t(y_1, B_\eta(\hat{x}^k)) \\
P_2^1(y_2, B_\eta(\hat{x}^k)) & \cdots & P_2^t(y_2, B_\eta(\hat{x}^k)) \\
P_3^1(y_3, B_\eta(\hat{x}^k)) & \cdots & P_3^t(y_3, B_\eta(\hat{x}^k)) \\
\vdots & & \vdots \\
P_m^1(y_m, B_\eta(\hat{x}^k)) & \cdots & P_m^t(y_m, B_\eta(\hat{x}^k))
\end{array}}^{t} \right.
\begin{array}{cc}
\rightarrow & \mu^1(B_\eta(\hat{x}^k)) \\
\rightarrow & \mu^2(B_\eta(\hat{x}^k)) \\
\rightarrow & \mu^3(B_\eta(\hat{x}^k)) \\
 & \vdots \\
\rightarrow & \mu^m(B_\eta(\hat{x}^k))
\end{array}
$$

$$
\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
1 & \cdots & 1 & & \beta
\end{array}
$$

But convergence in t is geometric and independent of m, i.e.,

$$|P_m^t(y_m, B_\eta(\hat{x}^k)) - \mu^m(B_\eta(\hat{x}^k))| \quad \leq \quad (1 - p_k^m)^t.$$

Outline

Conclusion

In one dimension, we have an asymptotic version of the median voter theorem that incorporates strategic decision making in a dynamic bargaining game.

In multiple dimensions, if the core is empty but the model is close to canonical, then long run equilibrium outcomes will be close to the core of the canonical model.

This extends the results of Ferejohn, McKelvey, and Packel (1984) to allow for farsighted agents and proposals generated by optimal equilibrium strategies.