DOMINANCE-BASED SOLUTIONS FOR STRATEGIC FORM GAMES

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November, 1997

INTRODUCTION

WE DO NOT ASSUME:

- players use mixed strategies/formulate probabilistic beliefs about opponents' strategies;
- expected utility maximization;
- common knowledge of mixed strategies;

WE DO ASSUME:

- players' decisions can be modeled by choice sets, rather than mixed strategies;
- common knowledge of choice sets, rather than of mixed strategies;
- choice sets adhere to some notion of dominance, which may or may not involve expected utility maximization.

WE GET:

- extension of the classical theory of choice/consumer theory to strategic situations;
- broader scope of application, including non-Bayesian players;
- generalization of Shapley's approach to two-player zero-sum games;
- unification of results on rationalizability;
- unified approach to the theory of tournaments.

SHAPLEY'S SADDLES

SET-UP:

- Let (X_1, X_2, u_1, u_2) be a finite two-player zero-sum game.

- A **generalized saddle** is a subset $Y_1 \times Y_2 \subseteq X_1 \times X_2$ such that, for each $x_1 \notin Y_1$, there exists $y_1 \in Y_1$ satisfying

$$(\forall x_2 \in Y_2)(u_1(y_1, x_2) > u_1(x_1, x_2)),$$

  and likewise for player 2.

- A **saddle** is a generalized saddle that is minimal with respect to set-inclusion: if $Z_1 \times Z_2$ is a generalized saddle with $Z_1 \subseteq Y_1$ and $Z_2 \subseteq Y_2$, then $Z_1 = Y_1$ and $Z_2 = Y_2$.

- A **generalized weak saddle** is a subset $Y_1 \times Y_2 \subseteq X_1 \times X_2$ such that, for each $x_1 \notin Y_1$, there exists $y_1 \in Y_1$ satisfying

$$(\forall x_2 \in Y_2)(u_1(y_1, x_2) \geq u_1(x_1, x_2))$$

  with strict inequality for some $x_2 \in Y_2$, and likewise for player 2.

- A **weak saddle** is a minimal generalized weak saddle.

SHAPLEY'S SADDLES (CONT.)

THEOREM (Shapley): There is exactly one saddle.

REMARKS:

- The weak saddle is not generally unique.
- The saddle possesses "internal" as well as "external" stability: if $Y_1 \times Y_2$ is a saddle, there do not exist distinct $x_1, y_1 \in Y_1$ such that

  $$(\forall x_2 \in Y_2)(u_1(y_1, x_2) > u_1(x_1, x_2)),$$

  and likewise for player 2.
- The same is not true of the weak saddle.

DIRECTIONS:

- multi-player, non-zero-sum games,
- non-minimal pairs $Y_1 \times Y_2$ possessing "external" and "internal" stability properties,
- other dominance concepts,
- infinite games.

STRATEGIC FORM GAMES

I	set of players (finite)
i, j	elements of I
X_i	i's strategy set (finite)
x_i, y_i, z_i	elements of X_i
$X = \Pi_{i \in I} X_i$	set of strategy profiles
x, y, z	elements of X
Y_i, Z_i	subsets of X_i
$Y = \Pi_{i \in I} Y_i$	product set
$Y_{-i} = \Pi_{j \neq i} Y_j$	partial product set
x_{-i}, y_{-i}, z_{-i}	elements of Y_{-i}
$u_i(x)$	i's payoff from x
Q_i	binary relation on X_i

FORMULATION OF CHOICE SETS

DEFINITION: A set Y_i possesses the **maximality property** with respect to Q_i if

$$(x_i \in Y_i) \Leftrightarrow (\neg \exists y_i \in X_i)(y_i Q_i x_i \wedge \neg x_i Q_i y_i).$$

DEFINITION:

(i) A set Y_i possesses the **inner solution property** with respect to Q_i if

$$(x_i \in Y_i) \Rightarrow (\forall y_i \in Y_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

(ii) A set Y_i possesses the **outer solution property** with respect to Q_i if

$$(x_i \notin Y_i) \Rightarrow (\exists y_i \in Y_i \setminus \{x_i\})(y_i Q_i x_i).$$

(iii) A set Y_i possesses the **solution property** with respect to Q_i if

$$(x_i \in Y_i) \Leftrightarrow (\forall y_i \in Y_i \setminus \{x_i\})(\neg y_i Q_i x_i).$$

PROPOSITION 1: If Q_i is transitive and irreflexive, then Y_i possesses the solution property with respect to Q_i if and only if it possesses the maximality property with respect to Q_i.

DEFINITION: A **dominance structure** is a mapping Q from players i and partial product sets Y_{-i} to binary relations $Q_i(Y_{-i})$ on X_i.

DEFINITION: A set Y is a Q-**solution** if, for all i, Y_i possesses the solution property with respect to $Q_i(Y_{-i})$.

DEFINITION: Let Q be a dominance structure.

(i) Q satisfies **irreflexivity** if, for all i and all Y_{-i}, $Q_i(Y_{-i})$ is irreflexive.

(ii) Q satisfies **transitivity** if, for all i and all Y_{-i}, $Q_i(Y_{-i})$ is transitive.

(iii) Q satisfies **monotonicity** if, for all i, all $x_i, y_i \in X_i$, and all $Y_{-i} \subseteq Z_{-i}$,

$$x_i Q_i(Z_{-i}) y_i \Rightarrow x_i Q_i(Y_{-i}) y_i.$$

PROPOSITION 2: Let Q be transitive. A set Y is a Q-solution if and only if

($\star$) for all i, Y_i is a minimal subset of X_i possessing the outer solution property with respect to $Q_i(Y_{-i})$.

SOME DOMINANCE STRUCTURES

Shapley Dominance

$$x_i S_i(Y_{-i}) y_i \quad \Leftrightarrow \quad (\forall x_{-i} \in Y_{-i})(u_i(x_i, x_{-i}) > u_i(y_i, x_{-i}))$$

Nash Dominance

$$x_i N_i(Y_{-i}) y_i \quad \Leftrightarrow \quad (\forall x_{-i} \in Y_{-i})(u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i}))$$

weak Shapley Dominance

$$x_i W_i(Y_{-i}) y_i \quad \Leftrightarrow \quad (x_i N_i(Y_{-i}) y_i) \wedge (\neg y_i N_i(Y_{-i}) x_i)$$

CLAIM:

- $\{x\}$ is a N-solution if and only if x is a Nash equilibrium.
- $\{x\}$ is a S-solution if and only if $\{x\}$ is a W-solution if and only if x is a strict Nash equilibrium.

CLAIM:

- The dominance structures N, W, and S are transitive. W and S are irreflexive, while N is not.
- The dominance structures S and N are monotonic, while W is not.

DOMINANCE STRUCTURES (CONT.)

EXAMPLE (distinct solutions):

	e	f	g
a	(1,-1)	(-1,1)	(0,0)
b	(-1,1)	(1,-1)	(0,0)
c	(-1,1)	(1,-1)	(0,0)
d	(-1,1)	(0,0)	(0,0)

CLAIM:

- $\{a, b\} \times \{e, f, g\}$ is a N-solution but not a W-solution or a S-solution.

- $\{a, b, c\} \times \{e, f, g\}$ is a W-solution but not a N-solution or a S-solution.

- $\{a, b, c, d\} \times \{e, f, g\}$ is a S-solution, but not a N-solution or a W-solution.

DEFINITION: A set Y is an **outer Q-solution** if, for all i, Y_i possesses the outer solution property with respect to $Q_i(Y_{-i})$.

DEFINITION: A set Y is **minimal** among a class of sets if, for all Z in that class, $Z \subseteq Y$ implies $Z = Y$. It is **maximal** if, for all Z in that class, $Z \supseteq Y$ implies $Z = Y$.

PROPOSITION 3: Let Q be transitive and monotonic. If Y is a minimal outer Q-solution then it is a (minimal) Q-solution.

PROOF: It suffices to show that a minimal outer Q-solution satisfies

> ($\star$) for all i, Y_i is a minimal subset of X_i possessing the outer solution property with respect to $Q_i(Y_{-i})$.

Suppose there is some i and $Z_i \subset\subset Y_i$ such that Z_i possesses the outer solution property with respect to $Q_i(Y_{-i})$. Consider $j \neq i$ and $x_j \notin Y_j$. The outer solution property with respect to $Q_j(Y_{-j})$ implies the existence of $y_j \in Y_j$ such that $y_j Q_j(Y_{-j}) x_j$. Then monotonicity implies $y_j Q_j(Z_i \times Y_{-i,j}) x_j$, and $Z_i \times Y_{-i}$ is an outer Q-solution, contradicting minimality of Y. //

EXISTENCE (CONT.)

PROPOSITION 4: If Q is transitive and monotonic then there is at least one Q-solution.

PROOF: X is an outer Q-solution. Since X is finite, a minimal outer Q-solution exists. //

COROLLARY 1: There exist a S-solution and a N-solution.

EXAMPLE (non-existence of W-solution):

	c	d
a	(2,1)	(1,2)
b	(1,2)	(1,1)

Note that $aW_1(\{c,d\})b$ and $dW_2(\{a\})c$, so player 2 cannot choose $\{c,d\}$; $aW_1(\{c\})b$ and $dW_2(\{a\})c$, so 2 cannot choose $\{c\}$; etc.

COMPARISONS OF Q-SOLUTIONS

DEFINITION: Q is **stronger** than Q' if, for all i, x_i, y_i, and Y_{-i}, $x_i Q_i(Y_{-i}) y_i \Rightarrow x_i Q'_i(Y_{-i}) y_i$.

RELATIONSHIPS FOR ABOVE Q'S:

PROPOSITION 5: Let Q be stronger than Q'.

(i) If Q' is transitive and monotonic then every Q-solution includes a Q'-solution.

(ii) If Q is transitive and monotonic then every Q'-solution is included in a Q-solution.*

RELATIONSHIPS FOR ABOVE Q'S:

COMPARISONS (CONT.)

EXAMPLE (N-solution vs. Nash equilibrium):

	c	d	e	f
a	(1,0)	(1,10)	(1,11)	(1,-1)
b	(1,10)	(1,0)	(1,-1)	(1,11)

The mixed strategies $(1/2, 1/2)$ and $(1/2, 1/2, 0, 0)$ form an equilibrium with minimal support, $\{a, b\} \times \{c, d\}$. But the only N-solutions are $\{a\} \times \{e\}$ and $\{b\} \times \{f\}$.

MIXED STRATEGY NASH EQUILIBRIUM

NOTATION: If $p = (p_i)_{i \in I}$ is a profile of mixed strategies, let

$$\sigma(p) = \{x \in X | \Pi_{i \in I} p_i(x_i) > 0\}$$

denote the support of p.

DEFINITION:

(i) A set Y_i possesses the **inner maximality property** with respect to Q_i if

$$(x_i \in Y_i) \Rightarrow (\neg \exists y_i \in X_i)(y_i Q_i x_i \wedge \neg x_i Q_i y_i).$$

(ii) A set Y is an **inner Q-maximum** if, for all i, Y_i possesses the inner maximality property with respect to $Q_i(Y_{-i})$.

NOTATION: Given a mixed strategy profile p, write $x_i Q_i^p(Y_{-i}) y_i$ if the expected payoff of x_i is greater than that of y_i, calculated with respect to $\Pi_i p_i$ conditional on Y.

THEOREM *: Mixed strategy profile p is an equilibrium if and only if $\sigma(p)$ is an inner Q^p-maximum.

THE RATIONALIZABILITY LITERATURE

DEFINITION:

(i) x_i is **ordinally rationalizable** if there is some monotonic transformation of u_i such that it is a best response to some profile of pure strategies; for each such strategy there is a monotonic transformation of payoffs that makes it a best response to some profile of pure strategies; and so on.

(ii) x_i is **correlated rationalizable** if it is a best response to some profile of (possibly correlated) mixed strategies; every strategy played with positive probability is a best response to a profile of (possibly correlated) mixed strategies; and so on.

(iii) x_i is **rationalizable** if it is a best response to some profile of (independent) mixed strategies; every strategy played with positive probability is a best response to a profile of (independent) mixed strategies; and so on.

(iv) x_i is **point rationalizable** if it is a best response to some profile of pure strategies; each such strategy is a best response to some profile of pure strategies; and so on.

RATIONALIZABILITY (CONT.)

PROPOSITION 6: Under each criterion, a set Y consists of the rationalizable strategy profiles if and only if it is the unique maximal solution for a corresponding dominance structure . . .

Börgers Dominance

$$x_i B_i(Y_{-i}) y_i \quad \Leftrightarrow \quad \text{for all } Z_{-i} \subseteq Y_{-i} \text{ there is some } z_i \text{ such that } x_i W_i(Z_{-i}) y_i.$$

Mixed Shapley Dominance

$$x_i S_i^*(Y_{-i}) y_i \quad \Leftrightarrow \quad y_i \text{ is strictly dominated over } Y_{-i} \text{ by some mixed strategy.}$$

Rationalizable Dominance

$$x_i R_i(Y_{-i}) y_i \quad \Leftrightarrow \quad y_i \text{ is a best response to no mixed strategy profile with support in } Y_{-i}.$$

Point Rationalizable Dominance

$$x_i P_i(Y_{-i}) y_i \quad \Leftrightarrow \quad y_i \text{ is a best response to no pure strategy profile in } Y_{-i}.$$

RATIONALIZABILITY (CONT.)

COMMON FEATURES:

- The rationalizable strategy profiles can be found by iterative deletion of dominated strategies.
- Order of elimination is irrelevant.

DEFINITION: Q is **weakly irreflexive** if, for all i, all $x_i, y_i \in X_i$, and all Y_{-i}, $x_i Q_i(Y_{-i}) x_i$ implies $y_i Q_i(Y_{-i}) x_i$.

PROPOSITION 6: If Q is weakly irreflexive, transitive, monotonic, and hard*, then

(i) the maximal Q-solution is unique;

(ii) it can be found by iterative deletion of Q-dominated strategies;

(iii) the order of elimination is irrelevant.

COROLLARY 2: There is exactly one maximal S-solution, the strategy profiles remaining after iterative deletion of strictly dominated strategies. Similarly for B, S^*, R, and P.

SHAPLEY SETS

DEFINITION: Y is a Q-**set** if it is a minimal Q-solution.

DEFINITION: A game is **equilibrium safe** if there exists a mixed strategy Nash equilibrium $p^* = (p_1^*, \ldots, p_n^*)$ such that, for all mixed strategy equilibria $p = (p_1, \ldots, p_n)$ and all i, p_i^* is a best response to p_{-i}.

CLAIM: A game is equilibrium safe if

- there is a unique mixed strategy Nash equilibrium;
- there is a dominant strategy equilibrium;
- mixed strategy equilibria are interchangeable;
- it is a two-player zero-sum game.

PROPOSITION 7:

(i) If the R-set is unique then so are the S-set and S^*-set.

(ii) In an equilibrium safe game, the R-set is unique.

EXTENSION: If a game is order equivalent to an equilibrium safe game, it has a unique S-set.

SHAPLEY SETS (CONT.)

EXAMPLE (order equivalence):

	a	b	c	d
a	(5,1)	(1,5)	(2,2)	(2,2)
b	(1,5)	(5,1)	(2,2)	(2,2)
c	(2,2)	(2,2)	(5,1)	(1,5)
d	(2,2)	(2,2)	(1,5)	(5,1)

This game has multiple R-sets and is not equilibrium safe, but it is order equivalent to an equilibrium safe game. (Change 2's to 3's.) Therefore, it has a unique S-set.

PROPOSITION 8: In a two-player game with weakly Pareto optimal payoffs, the S-set is unique.

APPLICATION TO TOURNAMENTS

SET-UP:

- Two parties, 1 and 2, choose policy platforms from a finite set, A.
- A vote between the two platforms is taken, the winner given by the majority relation, M.
- M is asymmetric and total: if $a \neq b$ then aMb or bMa.
- Parties care only about winning the election.

DEFINITION:

(i) A policy a is a **Condorcet winner** if, for all $b \neq a$, aMb.

(ii) A **topcycle set** is a minimal set B such that, for all $a \in B$ and all $b \notin B$, aMb.

(iii) For $B \subseteq A$ and $a, b \in B$, a **covers** b over B if aMb and, for all $c \in B$, bMc implies aMc. Let $UC(B)$ denote the elements of B that are not covered over B by any other elements of B.

(iv) B is a **covering set** if $UC(B) = B$ and, for all $a \in A \setminus B$, $a \notin UC(B \cup \{a\})$.

TOURNAMENTS (CONT.)

Q-SETS IN TOURNAMENTS:

	exist	unique max.	unique min.	tourn.
S	x	x	x	big
W				min. cov. set
N	x			min. cov. set
B	x	x	x	topcycle
S^*	x	x	x	topcycle
R	x	x	x	topcycle
P	x	x	x	topcycle

EXISTENCE IN INFINITE GAMES

FORMALITIES:

(i) Let N be an arbitrary set.

(ii) Let X_i be a compact space.

(iii) Let X and X_{-i} have the product topologies.

DEFINITION:

(i) A set Y is a Q-**solution** if, for all i, Y_i is a minimal compact subset of X_i possessing the outer solution property with respect to $Q_i(Y_{-i})$.

(ii) Q is **transitive-continuous** if, for all $i \in N$, all $x_i, y_i, z_i \in X_i$, all nets $x_i^\alpha \to x_i$, and all compact subsets $Y_{-i} \subseteq X_{-i}$,

$$(y_i Q_i(Y_{-i}) z_i) \wedge (\forall \alpha)(x_i^\alpha Q_i(Y_{-i}) y_i) \Rightarrow (x_i Q_i(Y_{-i}) z_i).$$

PROPOSITION 9: B, S^*, R, and P are transitive-continuous. If each u_i is upper semi-continuous in x_i then S, W, and N are transitive-continuous.

EXISTENCE (CONT.)

PROPOSITION 10: If Q is transitive-continuous and monotonic then there exists at least one Q-solution.

COROLLARY 3: There exist S-, N-, B-, S^*-, R-, and P-solutions.