Bargaining Foundations of the Median Voter Theorem

Seok-ju Cho
University of Rochester

John Duggan
University of Rochester

Brief introduction

- **Median Voter Theorem:** If an odd number of voters have single-peaked preferences over a unidimensional space, then the median ideal point is strictly majority-preferred to every other point.

- Non-cooperative underpinnings:

large groups	$\longrightarrow$	elections (Downs)
small groups	$\longrightarrow$	committees and legislatures (bargaining)

- Do bargaining equilibria lead to the median ideal point? If so, how generally? If not, how bad is it?

The bargaining literature

Parameters	Baron-Ferejohn	Banks-Duggan
players	$n > 2$	$n > 2$
alternatives	Δ	$\Re$
voting rule	M	M
proposer	random	random
status quo	bad	whatever

Results	Baron-Ferejohn	Banks-Duggan
stat. existence	Y	Y
stat. delay	N	N/Y
stat. uniqueness	Y	N
stat. mvt	n/a	Y
folk theorem	Y	?

Elements of the model

N	set of n agents (n odd)*
X	set of alternatives (a compact interval)
$u_i \colon X \to \Re$	stage utility (cont., strictly concave)*
$\tilde{x}_i$	i's ideal point (median is $\tilde{x}_k$)
q	status quo (in X)*
δ	discount factor (common, less than one)*
ρ_i	recognition prob. (positive, fixed)*

The bargaining protocol in period t

- an agent i is selected with probability ρ_i to propose, say, x

- each agent j votes accept (a) or reject (r)

- if $\#\{j \mid v_j = a\} > n/2$, then the game ends with outcome (x, t), and payoffs are

$$(1 - \delta)^{t-1} u_j(q) + \delta^{t-1} u_j(x)$$

- otherwise, the game continues to period $t + 1$ and is repeated

- if the game continues ad infinitum, then payoffs are $u_j(q)$

The bargaining protocol (cont.)

- Think of payoffs from (x, t) as the discounted sum of the flow of payoffs

$$\underbrace{u_i(q)\ \ u_i(q)\ \ \cdots\ \ u_i(q)}_{t-1 \text{ periods}}\ \ \underbrace{u_i(x)\ \ u_i(x)\ \ u_i(x)\cdots}_{\text{ad infinitum}}$$

Continuation lotteries

- A profile of strategies determines a probability distribution on histories, which determines a distribution, say λ, on outcomes.

- We refer to the distribution on outcomes following a rejected proposal as a "continuation lottery."

- We show $U_i = E_\lambda[u_i]$, where λ is the continuation lottery on X.

Voting model

- The usual approach:
  - voting is simultaneous
  - equilibria are stationary
  - weak dominance is applied

- The problem:
  - future play can be conditioned on individual votes
  - weak dominance loses its bite
  - too many Nash equilibria

Voting model (cont.)

- Our solution: Voting is sequential.
- The order of voting is determined randomly before and/or after the proposer is determined.
- Each order of voting is realized with a positive, fixed* probability.

Voting model (cont.)

- In stationary equilibria, the voting game reduces to a binary vote, and stage-game weak dominance is effective.

- Suppose x has been proposed.

	a	r
a	x	x
r	x	λ

a

	a	r
a	x	λ
r	λ	λ

r

  If row strictly prefers x to continuing, then voting r is weakly dominated in the stage game.

- In fact, x survives elimination if and only if it is weakly majority-preferred to continuing.

Voting model (cont.)

- Not so when stationarity is dropped.

	a	r
a	x	x
r	x	λ_1

a

	a	r
a	x	λ_2
r	λ_3	λ_4

r

- Suppose payoffs are as follows.

row	column	matrix
λ_1	λ_2	λ_3
x	x	x
λ_4	λ_4	λ_4
λ_2	λ_3	λ_1

Then λ_4 is a strict (therefore undominated) Nash equilibrium outcome, but all agents strictly prefer x.

Subgame perfect equilibria

- A strategy σ is a pure strategy subgame perfect equilibrium if there do not exist an agent, a history, and a deviation from σ that increases that agent's expected payoff following that history.

- Let $X(\delta)$ consist of alternatives x such that:

  there exists a proposer history h and a pure strategy subgame perfect equilibrium σ such that x passes with positive probability from h in σ.

- Let $V_i(\delta)$ consist of payoffs r such that:

  there exists a proposer history h and a pure strategy subgame perfect equilibrium σ such that i's expected payoff is r from h in σ.

Non-stationary equilibrium

- Example: three symmetric agents, the median proposes y or $-y$, right and left agents propose z and $-z$.

- As agents become patient, these proposals collapse to the median.

Non-stationary equilibrium (cont.)

- When the median proposes his/her ideal point, it is rejected.

Main result

- Given a sequence of sets $\{Y^m\}$ in Euclidean space and an element x, we write $Y^m \to x$ if $\sup_{y \in Y^m} ||y - x|| \to 0$.

- **Theorem:** Let $\{\delta^m\}$ be a sequence of discount factors converging to one. Then

  - $X(\delta^m) \to \tilde{x}_k$,

  - for all $i \in N$, $V_i(\delta^m) \to u_i(\tilde{x}_k)$.

- That is,

  - subgame perfect equilibrium outcomes converge to the median ideal point,

  - equilibrium delay becomes negligible.

Intuition for proof

- Suppose $X(\delta^m)$ does not converge to the median. Letting*

$$\begin{aligned} \underline{x}^m &= \min X(\delta^m) \\ \overline{x}^m &= \max X(\delta^m), \end{aligned}$$

  suppose $\underline{x}^m \to \underline{x}$ and $\overline{x}^m \to \overline{x}$.

- A typical situation is below.

- For each m, concavity implies that there is a majority C^m such that either $\underline{x}^m$ is the worst thing in the interval $[\underline{x}^m, \overline{x}^m]$ for every member of C^m or $\overline{x}^m$ is worst for all.

Intuition for proof (cont.)

- Lemma: Suppose $x \in X(\delta^m)$ is proposed in some equilibrium and passes with positive probability. Then there is an agent i^m in C^m who weakly prefers x to the continuation lottery from rejecting the proposal.

- Apply the lemma with $x = \overline{x}^m$ to get an agent i^m from each C^m who weakly prefers $\overline{x}^m$ to the continuation lottery from rejection. But...

- Lemma: In every subgame perfect equilibrium, if the median $\tilde{x}_k$ is proposed, then it passes with positive probability (bounded above zero).

Intuition for proof (cont.)

- As a consequence, agent i^m can obtain the median with positive probability when selected to propose.

- So why would the agent i^m be willing to vote for his/her worst alternative rather than reject it and at least obtain the median $\tilde{x}_k$ with positive probability?

- It must be that the status quo is inferior to $\overline{x}^m$, and that delay makes the agent worse off.

- But we show that, as the agent becomes patient, delay becomes inconsequential, a contradiction.

Extensions

- Weaken concavity: strict concavity can be replaced by piece-wise linear.

- Even number of agents: okay, but we need an agent to "break ties."

- Bad status quo: we can take the status quo out of X as long as every agent strictly prefers the median to the status quo.

- Heterogeneous discount factors: we just need discount factors to converge to one at a "uniform enough" rate, i.e.,

$$\frac{\ln(\delta_i^m)}{\ln(\delta_j^m)} \to 1$$

for all $i, j \in N$.

Extensions (cont.)

- Variable recognition probabilities: These can vary arbitrarily with histories, as long as each agent's probability is bounded strictly above zero.

- Variable probability of voting orders: Can vary arbitrarily with histories, as long as each order has probability bounded strictly above zero.

- Impromptu voting: We can draw agents to vote one at a time.

- Mixed strategy equilibria: We can allow agents to play any mixed strategy equilibria.

Quick conclusion

- In contrast to the distributive model of bargaining, an anti-folk theorem holds for the unidimensional model.

- The asymptotic median voter theorem for stationary equilibria extends very generally.

- Delay becomes negligible as agents become patient.

- The median voter theorem appears safe.