Bargaining Foundations of the Median Voter Theorem

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Outline of talk

- brief introduction
- bargaining literature
- model
- voting subgames
- main result
- intuition for proof
- extensions
- quick conclusion

Brief introduction

- The Median Voter Theorem: If an odd number of voters have single-peaked preferences over a unidimensional space, then the median ideal point is strictly majority-preferred to every other point.

- Non-cooperative underpinnings:

large groups	$\longrightarrow$	elections (Downs)
small groups	$\longrightarrow$	committees and legislatures (bargaining)

- Do bargaining equilibria lead to the median ideal point? If so, how generally?

The bargaining literature

Parameters	Rubinstein	Binmore
players	$n = 2$	$n = 2$
alternatives	$\Delta/\Re$	$\Delta/\Re$
voting rule	U/M	U/M
proposer	deterministic	random
status quo	bad	bad

Parameters	Baron-Ferejohn	Banks-Duggan
players	$n > 2$	$n > 2$
alternatives	Δ	$\Re$
voting rule	M	M
proposer	random	random
status quo	bad	whatever

The bargaining literature (cont.)

Issues	R	B	B-F	B-D
existence of stationary eq.	Y	Y	Y	Y
delay in stationary eq.	N	N	N	N*
uniqueness of stationary eq.	Y*	Y*	Y	N
asymptotic mvt	n/a	n/a	n/a	Y
folk thm	N*	N*	Y	

Elements of the model

N	set of n agents (n odd)*
X	set of alternatives (a compact interval)
$u_i\colon X \to \Re$	stage utility (cont., strictly concave)*
$\tilde{x}_i$	i's ideal point (median is $\tilde{x}_k$)
q	status quo (in X)*
δ	discount factor (common, less than one)*
ρ_i	recognition prob. (positive, fixed)*

The bargaining protocol in period t

- an agent i is selected with probability ρ_i to propose, say, x

- each agent j votes accept (a) or reject (r)

- if $\#\{j \mid v_j = a\} > n/2$, then the game ends with outcome (x, t), and payoffs are

$$(1-\delta)^{t-1} u_j(q) + \delta^{t-1} u_j(x)$$

- otherwise, the game continues to period $t+1$ and is repeated

- if the game continues ad infinitum, then payoffs are $u_j(q)$

The bargaining protocol (cont.)

- Think of payoffs from (x, t) as the discounted sum of the flow of payoffs

$$\underbrace{u_i(q)\ \ u_i(q)\ \ \cdots\ \ u_i(q)}_{t-1 \text{ periods}}\ \ \underbrace{u_i(x)\ \ u_i(x)\ \ u_i(x)\cdots}_{\text{ad infinitum}}$$

Continuation lotteries

- We can summarize future play by a "continuation lottery."

- For example, suppose in odd periods, there is a 1/3 chance that x passes and 2/3 chance of delay; in even periods, there is a 1/4 chance of x' and 3/4 chance of delay.

Continuation Lotteries (cont.)

- Then agent i's expected payoff U_i at the beginning of period 1 can be written as

$$U_i = \alpha u_i(x) + \beta u_i(x') + \gamma u_i(q)$$

where

$$\alpha = \frac{2}{6 - 3\delta^2}$$

$$\beta = \frac{\delta}{6 - 3\delta^2}$$

$$\gamma = \frac{4 - \delta - 3\delta^2}{6 - 3\delta^2}$$

and

$$\alpha + \beta + \gamma = 1.$$

- That is, $U_i = E_\lambda[u_i]$, where λ is the continuation lottery on X.

Voting subgames

- In stationary equilibria, the voting game reduces to a binary vote, and stage-game weak dominance is effective.

- Suppose x has been proposed.

	a	r
a	x	x
r	x	λ

a

	a	r
a	x	λ
r	λ	λ

r

  If row strictly prefers x to continuing, then voting r is weakly dominated in the stage game.

- In fact, x survives elimination if and only if it is weakly majority-preferred to continuing.

Voting subgames (cont.)

- Not so when stationarity is dropped.

	a	r
a	x	x
r	x	λ_1

a

	a	r
a	x	λ_2
r	λ_3	λ_4

r

- Suppose payoffs are as follows.

row	column	matrix
λ_1	λ_2	λ_3
x	x	x
λ_4	λ_4	λ_4
λ_2	λ_3	λ_1

Then λ_4 is a strict (therefore undominated) Nash equilibrium outcome, but all agents strictly prefer x.

Voting subgames (cont.)

- We model voting as sequential: this allows us to drop the stage-game weak dominance refinement and leaves stationary equilibrium outcomes unchanged.

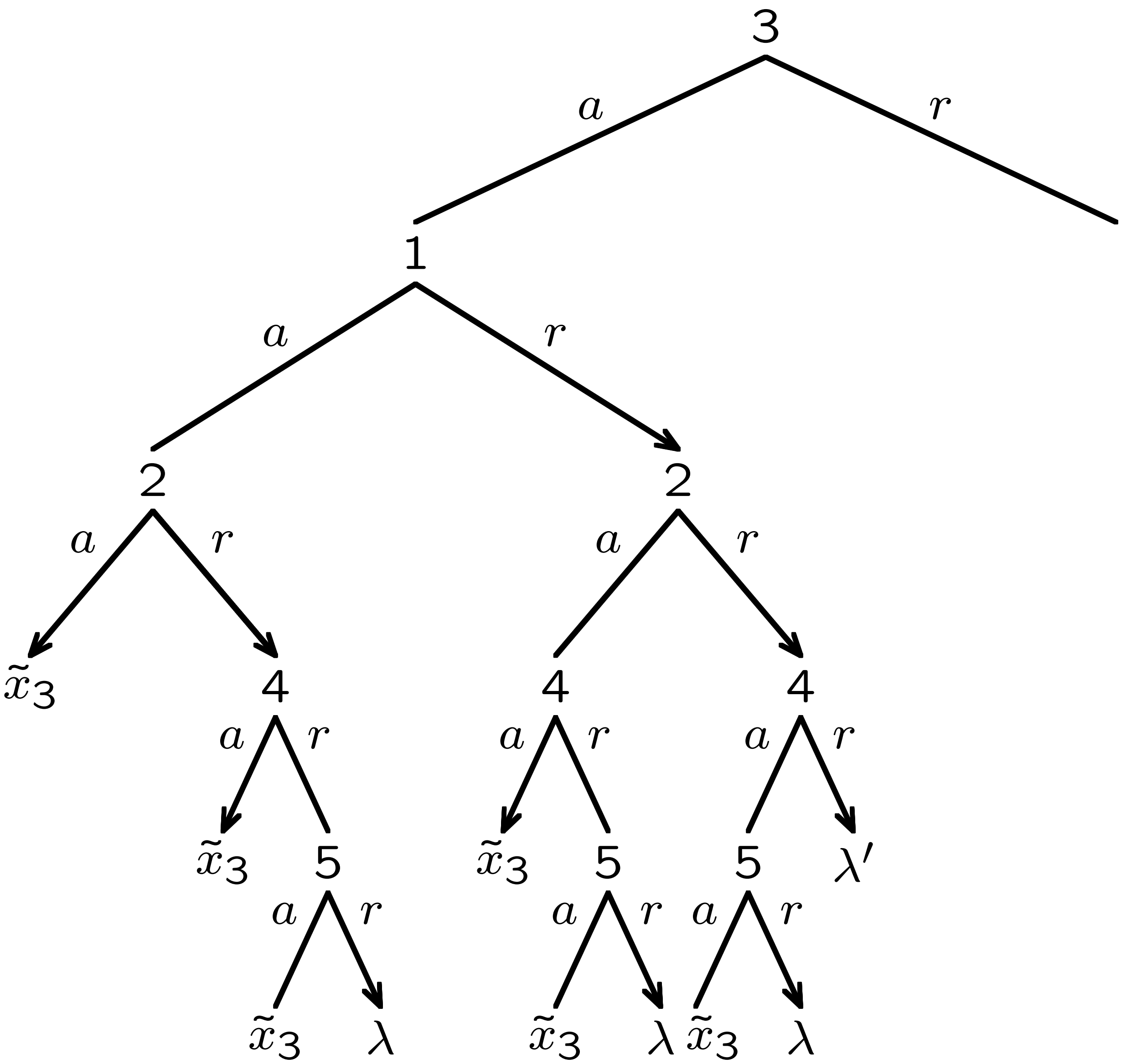
- A key property we want: the median $\tilde{x}_k$ passes if it is proposed.

- For this, we assume:

  - The order of voting is determined randomly before and/or after the proposer is determined.

  - With positive, fixed* probability, the median voter votes first, and voting then alternates from either side of the median.

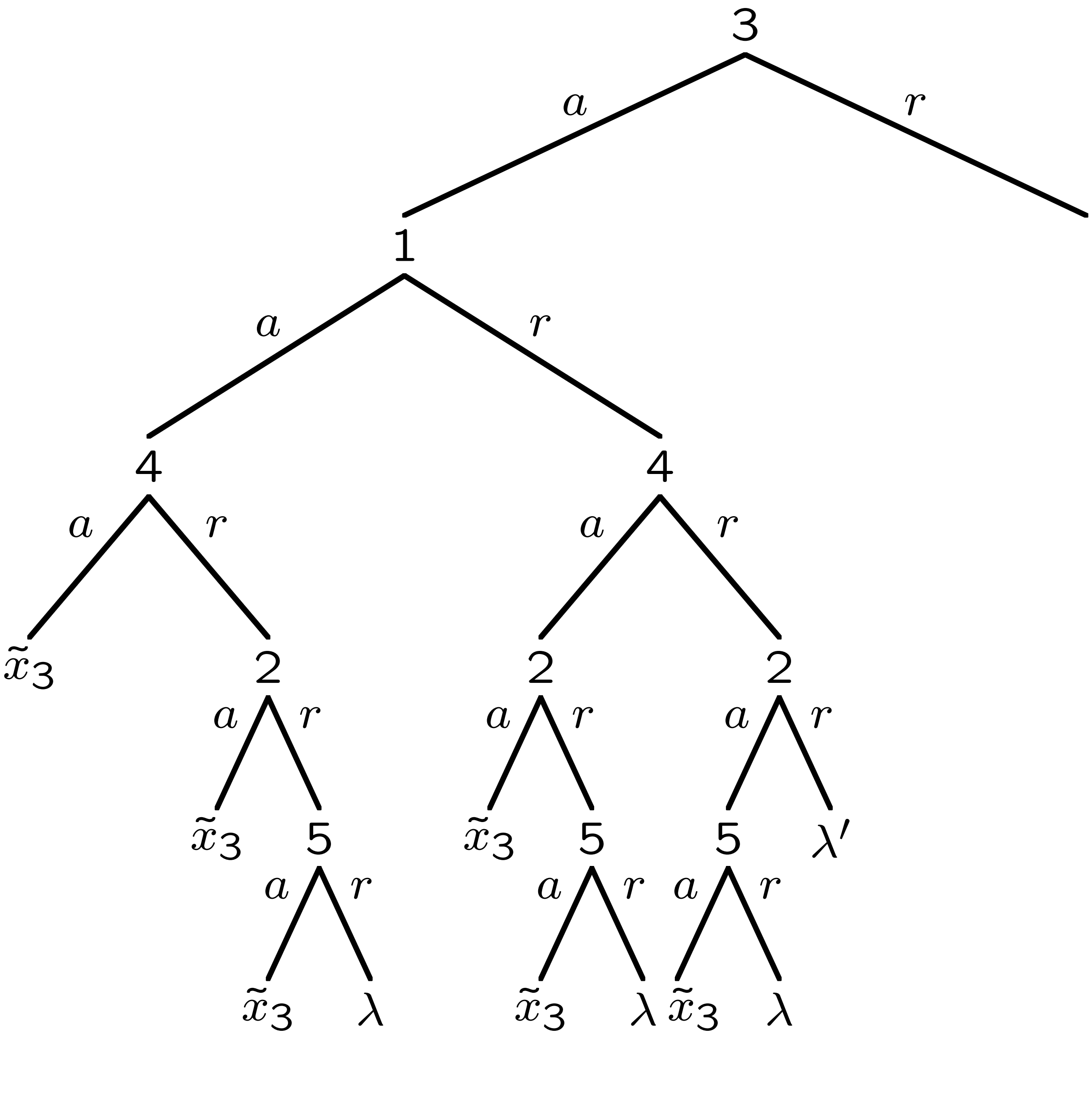
- The problem: Let ideal points of 1, 2, 3, 4, 5 be in order of voter indices. Suppose the median $\tilde{x}_3$ is proposed and 3 has voted a...



1	2	4	5
λ'	λ'	$\tilde{x}_3$	λ
$\tilde{x}_3$	$\tilde{x}_3$	λ'	$\tilde{x}_3$
		λ	

Voting subgames (cont.)

- Now suppose the voting order alternates. . .



Voting subgames (cont.)

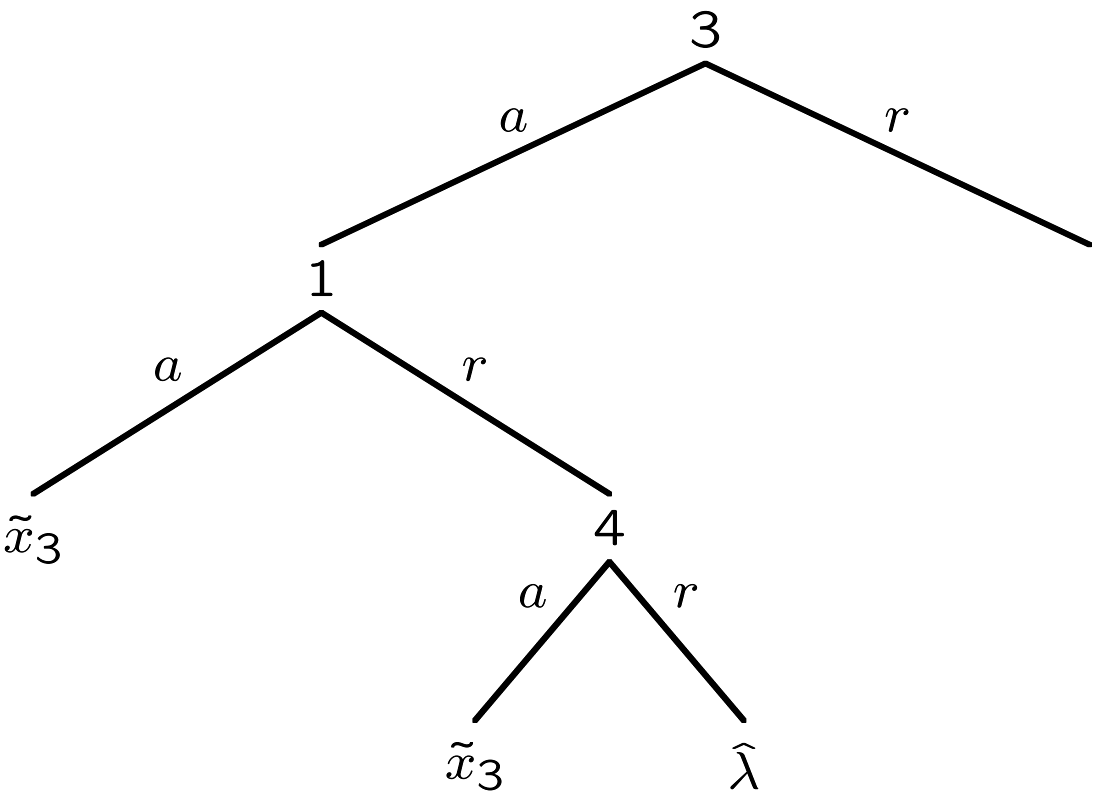
- By concavity, if agent 5 weakly prefers λ to the median $\tilde{x}_3$, and if agent 2 has the same weak preference, then the mean of λ must be equal to $\tilde{x}_3$.

- By strict concavity, the continuation lottery λ must have zero variance, so it is the point mass on $\tilde{x}_3$.

- Therefore, the only possible equilibrium outcome of the first two 2:5 voting subgames is the median.

Voting subgames (cont.)

- Then the game reduces to...



...and the same logic applies.

- Therefore, the median voter can (and will) obtain her ideal point by voting a in equilibrium.

Subgame perfect equilibria

- A strategy σ is a pure strategy subgame perfect equilibrium if there do not exist an agent, a history, and a deviation from σ that increases that agent's expected payoff following that history.

- Let $X(\delta)$ consist of alternatives x such that:

  there exists a proposer history h and a pure strategy subgame perfect equilibrium σ such that x passes with positive probability from h in σ.

- Let $V_i(\delta)$ consist of payoffs r such that:

  there exists a proposer history h and a pure strategy subgame perfect equilibrium σ such that i's expected payoff is r from h in σ.

The main result

- Given a sequence of sets $\{Y^m\}$ in Euclidean space and an element x, we write $Y^m \to x$ if $\sup_{y \in Y^m} ||y - x|| \to 0$.

- Theorem: Let $\{\delta^m\}$ be a sequence of discount factors converging to one. Then

  - $X(\delta^m) \to \tilde{x}_k$,

  - for all $i \in N$, $V_i(\delta^m) \to u_i(\tilde{x}_k)$.

- That is,

  - subgame perfect equilibrium outcomes converge to the median ideal point,

  - equilibrium delay becomes negligible.

Intuition for proof

- Suppose $X(\delta^m)$ does not converge to the median. Letting*

$$\begin{aligned} \underline{x}^m &= \min X(\delta^m) \\ \overline{x}^m &= \max X(\delta^m), \end{aligned}$$

  suppose $\underline{x}^m \to \underline{x}$ and $\overline{x}^m \to \overline{x}$.

- A typical situation is below.

- For each m, concavity implies that there is a majority C^m such that either $\underline{x}^m$ is the worst thing in the interval $[\underline{x}^m, \overline{x}^m]$ for every member of C^m or $\overline{x}^m$ is worst for all.

Intuition for proof (cont.)

- Lemma: Suppose $x \in X(\delta^m)$ is proposed in some equilibrium and passes with positive probability. Then there is an agent i^m in C^m who weakly prefers x to the continuation lottery from rejecting the proposal.

- Apply the lemma with $x = \overline{x}^m$ to get an agent i^m from each C^m who weakly prefers $\overline{x}^m$ to the continuation lottery from rejection. But. . .

- Lemma: In every subgame perfect equilibrium, if the median $\tilde{x}_k$ is proposed, then it passes with positive probability (bounded above zero).

Intuition for proof (cont.)

- As a consequence, agent i^m can obtain the median with positive probability when selected to propose.

- So why would the agent i^m be willing to vote for his/her worst alternative rather than reject it and at least obtain the median $\tilde{x}_k$ with positive probability?

- It must be that the status quo is inferior to $\overline{x}^m$, and that delay makes the agent worse off.

- But we show that, as the agent becomes patient, delay becomes inconsequential, a contradiction.

Extensions

- Even number of agents: okay, but we need one of the median voters to "break ties."

- Concavity: strict concavity can be weakened to allow for piece-wise linear.

- Bad status quo: we can take the status quo out of X as long as every agent strictly prefers the median to the status quo.

- Heterogeneous discount factors: we just need discount factors to converge to one at a "uniform enough" rate, i.e.,

$$\frac{\ln(\delta_i^m)}{\ln(\delta_j^m)} \to 1$$

for all $i, j \in N$.

Extensions (cont.)

- Variable recognition probabilities: These can vary arbitrarily with histories, as long as each agent's probability is bounded strictly above zero.

- Variable probability of alternating voting order: Can vary arbitrarily with histories, as long as voting alternates with probability bounded strictly above zero.

- Mixed strategy equilibria: mixed proposal strategies are not a problem; mixed voting strategies introduce some complications, but these should not be a problem.

Quick conclusion

- In contrast to the distributive model of bargaining, an anti-folk theorem holds for the unidimensional model.
- The asymptotic median voter theorem for stationary equilibria extends quite generally.
- Delay becomes negligible as agents become patient.
- The median voter theorem for small groups is pretty "safe."