Bargaining Foundations of the Median Voter Theorem

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Outline of talk

- social choice approach
- bargaining model
- stationary equilibria
  - existence, continuity, delay, core equivalence in one dimension
- voting subgames
- main result
- intuition for proof
- technical point
- extensions
- quick conclusion

Social choice approach

N	set of n agents (n odd)*
$X \subseteq \Re^m$	set of alternatives (compact, convex)
$u_i\colon X \to \Re$	i's utility function (cont., strictly concave)*
$\tilde{x}_i$	i's ideal point
$\mathcal{M}$	collection of majority coalitions
M	majority dominance relation: xMy means that $\{i \mid u_i(x) > u_i(y)\} \in \mathcal{M}$

The Social choice approach (cont.)

- The **core**, denoted K, consists of the "unbeaten" alternatives: $x \in K$ if and only if there is no $y \in X$ such that yMx.

- Under our maintained assumptions, if the core is non-empty, then it is a singleton.

- **Median Voter Theorem:** Assume $X \subseteq \Re$, and let k satisfy

$$\#\{i \mid \tilde{x}_i < \tilde{x}_k\} < \frac{n}{2}$$

and

$$\#\{i \mid \tilde{x}_i > \tilde{x}_k\} < \frac{n}{2}.$$

Then $K = \{\tilde{x}_k\}$. Moreover, for every $y \in X \setminus K$, $\tilde{x}_k M y$.

Social choice approach (cont.)

- Picture of median voter theorem:

- Problems in multiple dimensions:

The goal

- We want a general non-cooperative theory of collective decision-making that generates equilibrium predictions even in multiple dimensions.

- Even better if consistent with the median voter theorem in one dimension.
  - characterization of equilibrium outcomes
  - non-cooperative underpinning of cooperative solution

Bargaining literature

- Rubinstein (1982)
- Binmore (1987)
- Baron and Ferejohn (1989)
- Banks and Duggan (2000, 2003)

Bargaining framework

N	set of n agents (n odd)*
$X \subseteq \Re^m$	set of alternatives (compact, convex)
$u_i\colon X \to \Re$	i's stage utility function (cont., strictly concave)*
q	status quo
δ	discount factor (common, less than one)*
ρ_i	recognition probability (positive, fixed)*

Bargaining protocol in period t

- an agent i is selected with probability ρ_i to propose, say, x

- each agent j votes accept (a) or reject (r)

- if $\{j \mid v_j = a\} \in \mathcal{M}$, then the game ends with outcome (x, t), and payoffs are

$$(1-\delta)^{t-1} u_j(q) + \delta^{t-1} u_j(x)$$

- otherwise, the game continues to period $t+1$ and is repeated

- if the game continues ad infinitum, then payoffs are $u_j(q)$

Bargaining protocol (cont.)

- Think of payoffs from (x,t) as the discounted sum of the flow of payoffs

$$\underbrace{u_i(q)\ \ u_i(q)\ \ \cdots\ \ u_i(q)}_{t-1 \text{ periods}}\ \ \underbrace{u_i(x)\ \ u_i(x)\ \ u_i(x)\cdots}_{\text{ad infinitum}}$$

Two models of status quo

(A) the status quo is just an alternative, $q \in X$

(B) the status quo is "bad," i.e., for all $i \in N$ and all $x \in X$, $u_i(x) \geq u_i(q)$.

Stationary strategies

$p_i \in X$	pure proposal strat.
$\pi_i \in \mathcal{P}(X)$	mixed proposal strat.
$A_i \subseteq X$	acceptance set
$A_C = \bigcap_{i \in C} A_i$	accept. set for C
$A = \bigcup_{C \in \mathcal{M}} A_C$	social accept. set
$v_i(\sigma)$	continuation value
$(1-\delta)u_i(q) + \delta v_i(\sigma)$	reservation value

Stationary strategies (cont.)

- We say σ exhibits **delay** if, for some $i \in N$,

$$\pi_i(X \setminus A) > 0.$$

  Otherwise, it is **no delay.**

- Under (A), we say σ is **static** if, for all $i \in N$,

$$\pi_i(A \setminus \{q\}) = 0.$$

Stationary equilibria

- Voting strategies are stage-game undominated: for all $i \in N$,

$$A_i^* \;=\; \{x \mid u_i(x) \geq (1-\delta)u_i(q) + \delta v_i(\sigma^*)\}.$$

- Proposal strategies are optimal: for all $i \in N$,

$$\pi_i^*(\arg\max\{u_i(y) \mid y \in A^*\}) \;=\; 1$$

whenever

$$\begin{aligned} &\sup\{u_i(y) \mid y \in A^*\} \\ &\quad > \;(1-\delta)u_i(q) + \delta v_i(\sigma^*), \end{aligned}$$

and so on.

One-dimensional example

- Let $n = 3$, even chance recognition probabilities, quadratic utilities.

$$\Delta \;=\; \sqrt{\frac{(1-\delta)(\tilde{x}_2 - q)^2}{1 - 2\delta/3}}.$$

- Note
  - unique stationary equilibrium, no-delay, pure strategies
  - $\delta \to 1$ implies $\Delta \to 0$
  - $q \to \tilde{x}_2$ implies $\Delta \to 0$

Existence and continuity

- **Theorem:** There exists a no-delay stationary equilibrium.

- **Theorem:** If $X \subseteq \Re$, then every no-delay stationary equilibrium is in pure strategies.

- **Theorem:** The correspondence of no-delay stationary equilibrium mixed proposal strategies is upper hemicontinuous in the parameters of the model.

Delay

- **Theorem:** Under (B), all stationary equilibria are no-delay.

- Remark: The above result does not rely on strict concavity.

- **Theorem:** Assume $\delta > 0$. Under (A), if σ is a stationary equilibrium, then σ is static or no-delay.

- Remark: Under (A), delay may occur if $\delta = 0$ or if agents are risk neutral.

Delay (cont.)

- **Theorem:** Under (A), there exists a static stationary equilibrium if and only if $q \in K$.

- Thus,

  - when the core is empty (in multiple dimensions), every stationary equilibrium is no-delay.

  - in one dimension, unless $q \in K$, no stationary equilibrium is static, so all stationary equilibria are no-delay.

- **Theorem:** Assume $X \subseteq \Re$ and $\delta > 0$. If $q \notin K$, then $q \notin A^*$ in every stationary equilibrium.

Uniqueness in one dimension

- **Theorem:** Assume $X \subseteq \Re$ and each u_i is quadratic. There is a unique no-delay stationary equilibrium.

- **Theorem:** Assume $X \subseteq \Re$. If $\hat{\sigma}$ and $\tilde{\sigma}$ are stationary equilibria, then either $\hat{A} \subseteq \tilde{A}$ or $\tilde{A} \subseteq \hat{A}$.

- Thus, there is a unique minimal and a unique maximal no-delay stationary equilibrium.

- Remark: With non-quadratic utilities, there may be multiple no-delay stationary equilibria.

Core equivalence in one dimension

- **Asymptotic Median Voter Theorem I:** Assume $X \subseteq \Re$. Let $\delta^m \to 1$, and let $\{\sigma^m\}$ be a sequence of stationary equilibria. Then $A^m \to K$.

- **Asymptotic Median Voter Theorem II:** Assume $X \subseteq \Re$ and $\delta > 0$. Let σ^* be any stationary equilibrium. Under (A),

  - if $\{q\} \neq K$, then $q \notin A^*$

  - if $\{q\} = K$, then $A^* = \{q\}$, so the status quo is the unique stationary equilibrium outcome

  - if $q \to K$, then $A^* \to K$.

Folk theorems in bargaining

- Let X be the unit simplex in $\Re^n$, so $x = (x_1, \ldots, x_n)$ is an allocation of a "dollar" to the agents. Let $u_i(x) = x_i$. Then every allocation can be supported as a subgame perfect equilibrium outcome when the agents are patient enough in. . .
  - the multi-agent Rubinstein bargaining model
  - the Baron-Ferejohn model of majority bargaining.

- Is stationarity critical for the asymptotic median voter results? Is there a folk theorem for the one-dimensional bargaining model?

Continuation lotteries

- We can summarize future play by a "continuation lottery."

- For example, suppose in odd periods, there is a 1/3 chance that x passes and 2/3 chance of delay; in even periods, there is a 1/4 chance of x' and 3/4 chance of delay.

- Then agent i's expected payoff U_i at the beginning of period 1 satisfies:

$$\frac{U_i}{1-\delta} = \frac{1}{3}\left[u_i(x) + \delta u_i(x) + \cdots\right] + \frac{2}{3}\left[u_i(q) + \frac{\delta}{4}\left[u_i(x') + \delta u_i(x') + \cdots\right] + \frac{3\delta}{4}\left[u_i(q) + \frac{\delta U_i}{1-\delta}\right]\right]$$

Continuation lotteries (cont.)

- Then agent i's expected payoff U_i at the beginning of period 1 can be written as

$$U_i \;=\; \alpha u_i(x) + \beta u_i(x') + \gamma u_i(q)$$

where

$$\alpha = \frac{2}{6-3\delta^2}$$
$$\beta = \frac{\delta}{6-3\delta^2}$$
$$\gamma = \frac{4-\delta-3\delta^2}{6-3\delta^2}$$

and

$$\alpha + \beta + \gamma \;=\; 1.$$

- That is, $U_i = E_\lambda[u_i]$, where λ is the continuation lottery on X.

Voting subgames

- In stationary equilibria, the voting game reduces to a binary vote, and stage-game weak dominance is effective.

- Suppose x has been proposed.

a

	a	r
a	x	x
r	x	λ

r

	a	r
a	x	λ
r	λ	λ

If row strictly prefers x to continuing, then voting r is weakly dominated in the stage game.

- In fact, x survives elimination if and only if it is weakly majority-preferred to continuing.

Voting subgames (cont.)

- Not so when stationarity is dropped.

	a	r
a	x	x
r	x	λ_1

a

	a	r
a	x	λ_2
r	λ_3	λ_4

r

- Suppose payoffs are as follows.

row	column	matrix
λ_1	λ_2	λ_3
x	x	x
λ_4	λ_4	λ_4
λ_2	λ_3	λ_1

Then λ_4 is a strict (therefore undominated) Nash equilibrium outcome, but all agents strictly prefer x.

Voting subgames (cont.)

- We model voting as **sequential:** this allows us to drop the stage-game weak dominance refinement and leaves stationary equilibrium outcomes unchanged.

- A **key property** we want: the median $\tilde{x}_k$ passes with positive probability if it is proposed.

- For this, we assume:

  - The order of voting is determined randomly after the proposer is determined.

  - Every order of voting has a positive, fixed* probability.

Subgame perfect equilibria

- A strategy σ is a pure strategy subgame perfect equilibrium if there do not exist an agent, a history, and a deviation from σ that increases that agent's expected payoff following that history.

- Let $X(\delta)$ consist of alternatives x such that:

  there exists a proposer history h and a pure strategy subgame perfect equilibrium σ such that x passes with positive probability from h in σ.

- Let $V_i(\delta)$ consist of payoffs r such that:

  there exists a proposer history h and a pure strategy subgame perfect equilibrium σ such that i's expected payoff is r from h in σ.

Main result

- Given a sequence of sets $\{Y^m\}$ in Euclidean space and an element x, we write $Y^m \to x$ if $\sup_{y \in Y^m} ||y - x|| \to 0$.

- **Asymptotic MVT I:** Let $\{\delta^m\}$ be a sequence of discount factors converging to one. Then

  - $X(\delta^m) \to \tilde{x}_k$,

  - for all $i \in N$, $V_i(\delta^m) \to u_i(\tilde{x}_k)$.

- That is,

  - subgame perfect equilibrium outcomes converge to the median ideal point,

  - equilibrium delay becomes negligible.

Intuition for proof

- Suppose $X(\delta^m)$ does not converge to the median. Letting*

$$\begin{aligned}\underline{x}^m &= \min X(\delta^m)\\ \overline{x}^m &= \max X(\delta^m),\end{aligned}$$

suppose $\underline{x}^m \rightarrow \underline{x}$ and $\overline{x}^m \rightarrow \overline{x}$.

- A typical situation is below.

- For each m, concavity implies that there is a majority C^m such that either $\underline{x}^m$ is the worst thing in the interval $[\underline{x}^m, \overline{x}^m]$ for every member of C^m or $\overline{x}^m$ is worst for all.

Intuition for proof (cont.)

- **Lemma:** Suppose $x \in X(\delta^m)$ is proposed in some equilibrium and passes with positive probability. Then there is an agent i^m in C^m who weakly prefers x to the continuation lottery from rejecting the proposal.

- Apply the lemma with $x = \overline{x}^m$ to get an agent i^m from each C^m who weakly prefers $\overline{x}^m$ to the continuation lottery from rejection. But. . .

- **Lemma:** In every subgame perfect equilibrium, if the median $\tilde{x}_k$ is proposed, then it passes with positive probability (bounded above zero).

Intuition for proof (cont.)

- As a consequence, agent i^m can obtain the median with positive probability when selected to propose.

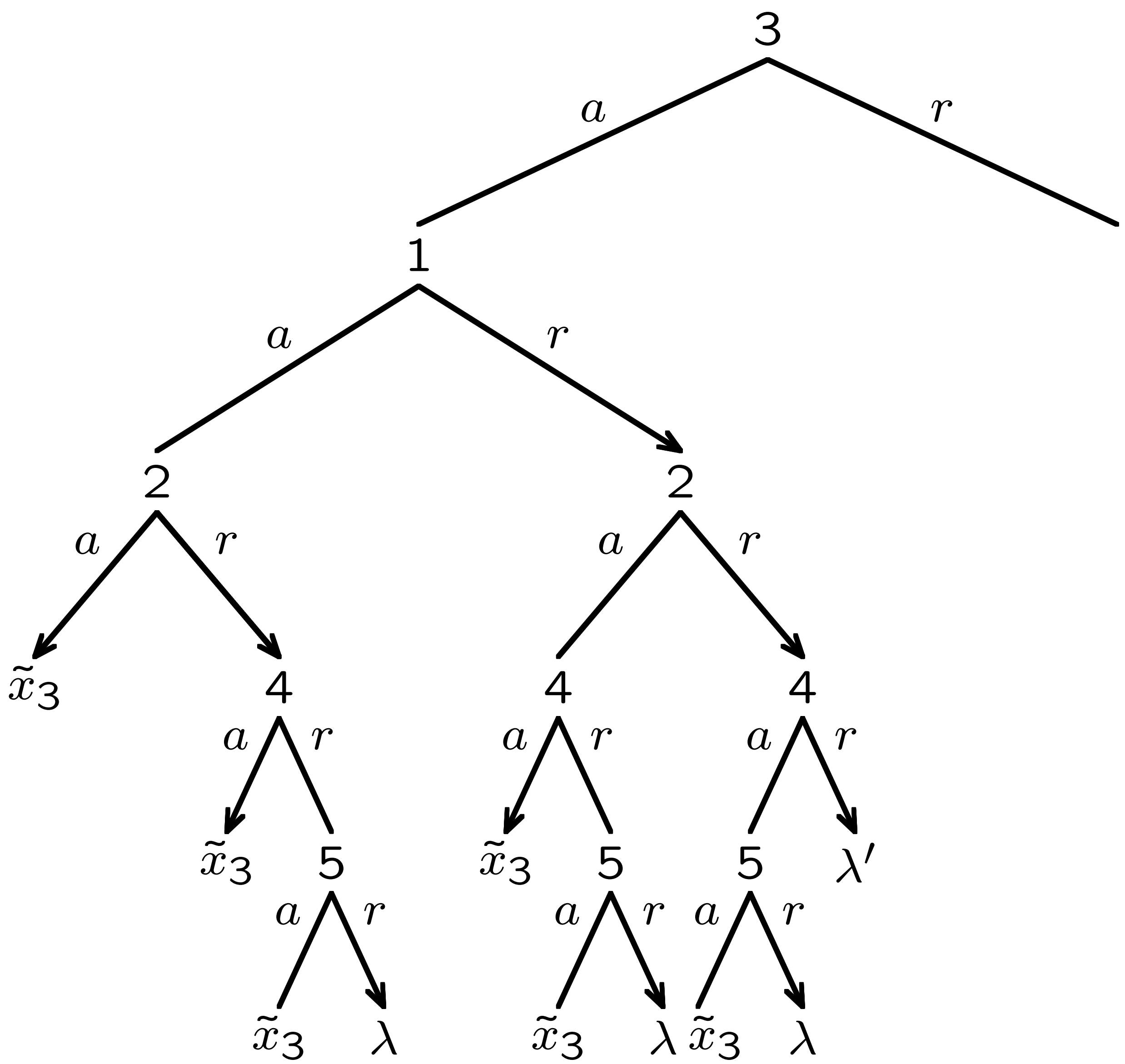
- So why would the agent i^m be willing to vote for his/her worst alternative rather than reject it and at least obtain the median $\tilde{x}_k$ with positive probability?

- It must be that the status quo is inferior to $\overline{x}^m$, and that delay makes the agent worse off.

- But we show that, as the agent becomes patient, delay becomes inconsequential, a contradiction.

Subtle technical point

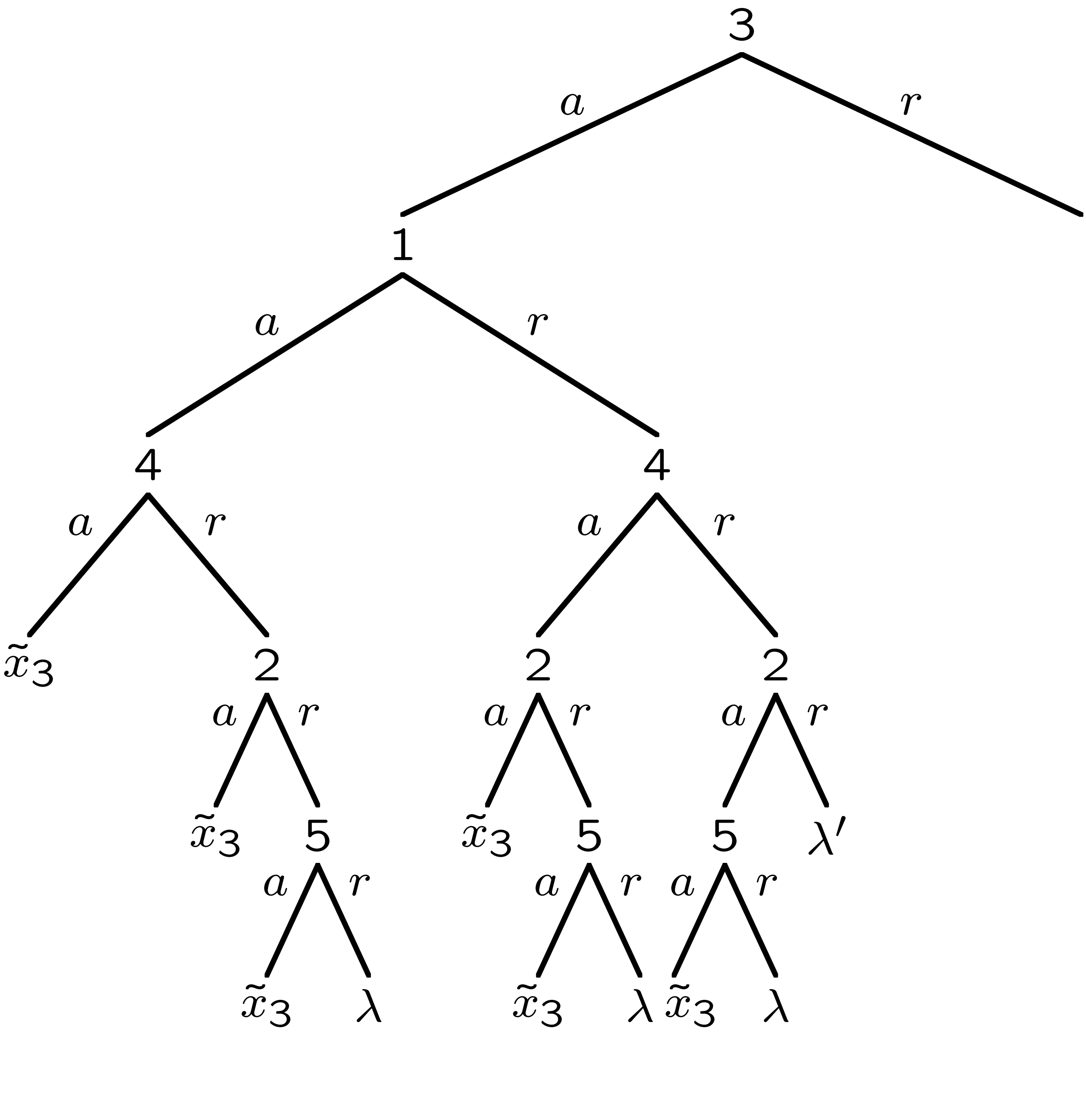
- Let ideal points of 1, 2, 3, 4, 5 be in order of voter indices. Suppose the median $\tilde{x}_3$ is proposed and 3 has voted a...



1	2	4	5
λ'	λ'	$\tilde{x}_3$	λ
$\tilde{x}_3$	$\tilde{x}_3$	λ'	$\tilde{x}_3$
		λ	

Technicality (cont.)

- Now suppose the voting order alternates. . .



Technicality (cont.)

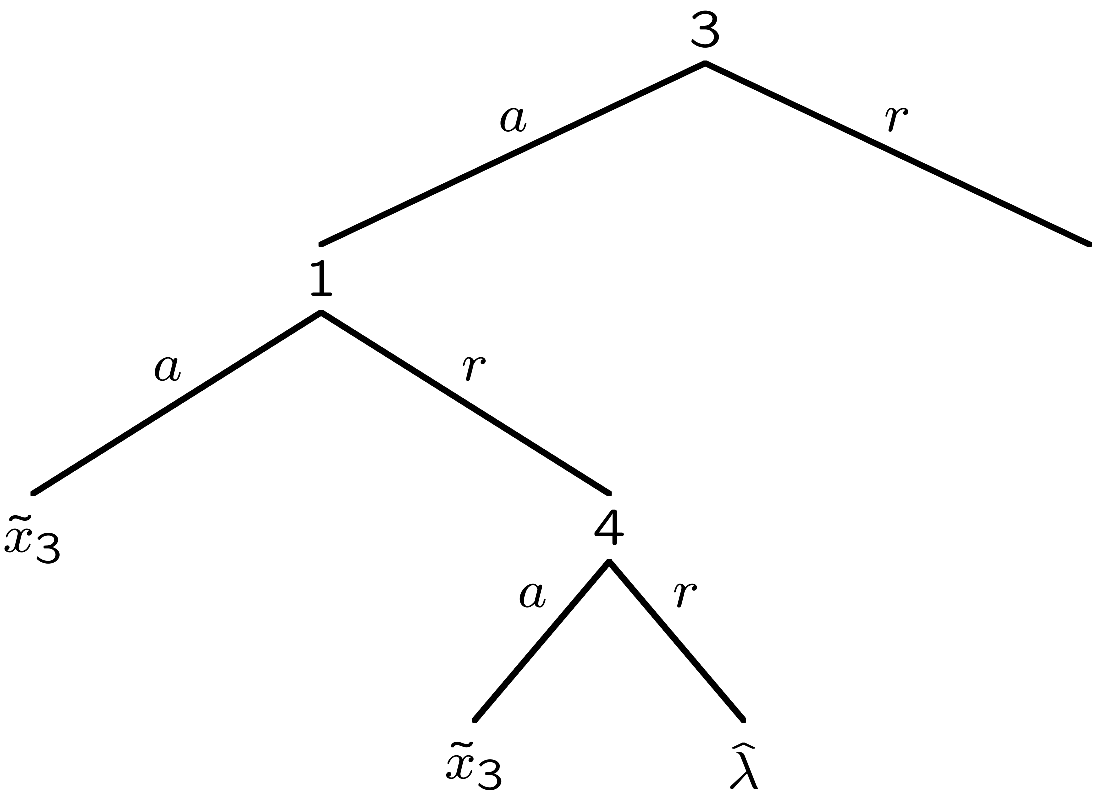
- By concavity, if agent 5 weakly prefers λ to the median $\tilde{x}_3$, and if agent 2 has the same weak preference, then the mean of λ must be equal to $\tilde{x}_3$.

- By strict concavity, the continuation lottery λ must have zero variance, so it is the point mass on $\tilde{x}_3$.

- Therefore, the only possible equilibrium outcome of the first two 2:5 voting subgames is the median.

Technicality (cont.)

- Then the game reduces to. . .



. . . and the same logic applies.

- Therefore, the median voter can (and will) obtain her ideal point by voting a in equilibrium.

Extensions of Main Result

- Even number of agents: okay, but we need one of the median voters to "break ties."

- Concavity: strict concavity can be weakened to allow for piece-wise linear.

- Heterogeneous discount factors: we just need discount factors to converge to one at a "uniform enough" rate, i.e.,

$$\frac{\ln(\delta_i^m)}{\ln(\delta_j^m)} \to 1$$

  for all $i, j \in N$.

Extensions (cont.)

- Variable recognition probabilities: These can vary arbitrarily with histories, as long as each agent's probability is bounded strictly above zero.

- Variable probability of voting orders: Can vary arbitrarily with histories, as long as voting alternates with probability bounded strictly above zero.

- Mixed strategy equilibria: mixed proposal strategies are not a problem; mixed voting strategies introduce some difficulties.

- Asymptotic MVT II. . .

Quick conclusion

- In contrast to the “divide the dollar” model of bargaining, an anti-folk theorem holds for the one-dimensional model.

- The asymptotic MVT I for stationary equilibria extends quite generally.

- Delay becomes negligible as agents become patient.