Electoral Competition with Policy-motivated Candidates

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Introduction

- Models of elections assume candidate commitment (Downs) or not (citizen candidates, electoral accountability).

- Advantages of commitment: simple model of campaigns, works in one dimension.

- Disadvantage: equilibrium existence problem with multiple policy dimensions

- Salvaging commitment?
  - mixed strategies
  - repeated elections
  - probabilistic voting
  - policy-motivated candidates

The Environment

A, B	candidates (C)
N	set of n voters (i, j, etc.)
$X \subseteq \Re^d$	convex policy space
$u_i \colon X \to \Re$	voter i's utility function
P	strict majority preference
R	weak majority preference
I	majority indifference
$u_C \colon X \to \Re$	candidate C's utility function

The Environment (cont.)

- Assumptions on utilities:
  - strictly concave,
  - $\tilde{x}_i = \arg\max u_i$,
  - distinct ideal points.

- Social preferences:

$$xPy \;\Leftrightarrow\; \#\{i \mid u_i(x) > u_i(y)\} \;>\; \frac{n}{2}$$

$$xRy \;\Leftrightarrow\; \#\{i \mid u_i(x) \geq u_i(y)\} \;\geq\; \frac{n}{2}$$

$$xIy \;\Leftrightarrow\; xRy \text{ and } yRx.$$

Social Choice

- Say x is a *core point* if, for all $y \in X$, we have xRy. The *core* is the set of all core points.

- The core may be empty.

- If nonempty and n is odd, then the core consists of a single point, x^*.

- Then $x^* = \tilde{x}_k$ for some voter k, and Plott's symmetry condition holds: for all $p \in \Re^d$,

$$\begin{aligned} &\#\{i \mid \exists \alpha > 0 \text{ s.t. } \nabla u_i(x^*) = \alpha p\} \\ &= \#\{i \mid \exists \alpha < 0 \text{ s.t. } \nabla u_i(x^*) = \alpha p\}. \end{aligned}$$

Social Choice (cont.)

- A picture of symmetry:

- If any voter's gradient is perturbed, then x^* ceases to be a core point.

- Generically, when n is odd and $d \geq 2$, the core is empty.

- When n is even, things are almost as bad.

The Electoral Game

- Candidates simultaneously choose policy platforms, $x_A, x_B \in X$.

- Voters vote sincerely, eliminating weakly dominated strategies. No abstention. Indifferences and ties broken randomly, e.g.,

$$\text{Pr}_A(x_A, x_B) = \begin{cases} 1 & \text{if } x_A P x_B \\ \frac{1}{2} & \text{if } x_A I x_B \\ 0 & \text{if } x_B P x_A. \end{cases}$$

- Office motivation:

$$U_C(x_A, x_B) = \text{Pr}_C(x_A, x_B).$$

- Policy motivation:

$$\begin{aligned} U_C(x_A, x_B) &= \text{Pr}_A(x_A, x_B) u_C(x_A) \\ &\quad + \text{Pr}_B(x_A, x_B) u_C(x_B). \end{aligned}$$

Results for Office Motivation

- (x_A, x_B) is a Nash equilibrium if and only if x_A and x_B are core points.

- In one dimension,

- In multiple dimensions,
  - n odd $\Rightarrow$ generic nonexistence
  - n even $\Rightarrow$ almost as bad.

Results for Policy Motivation

- In one dimension, if candidates' ideal points are on opposite sides of the median, and if n is odd, then the unique equilibrium has both candidates at the median.

- n even?

- In multiple dimensions, the only result we know of assumes:

  - n odd,

  - nonempty core,

  - Euclidean preferences,

  - candidates' gradients are linearly independent.

Equilibrium Typology

- Say (x_A, x_B) is *nonsatiated* if

$$\nabla u_A(x_A) \neq 0 \text{ and } \nabla u_B(x_B) \neq 0.$$

- It is *nonaligned* if the gradients do not point in the same direction: there do not exist $\alpha, \beta \geq 0$ (at least one nonzero) such that

$$\alpha \nabla u_A(x_A) = \beta \nabla u_B(x_B).$$

- Note: nonaligned $\Rightarrow$ nonsatiated.

Equilibrium Typology (cont.)

- Nonsatiated and nonaligned
- Nonsatiated and nonaligned
- Nonsatiated but not nonaligned
- Neither nonsatiated nor nonaligned

Examples

- We may have nonaligned equilibria, even when the core is empty.

Examples (cont.)

- We may have nonaligned equilibria not at the core, even when the core is nonempty.

- Note:
  - These equilibria are robust.
  - Policy platforms coincide.
  - No voters' gradients lie in the cones.

Necessary Conditions for Equilibrium

- **Theorem 1:** If (x_A, x_B) is a nonsatiated equilibrium, then $x_A = x_B$.

- Sketch for n odd: Suppose $x_A \neq x_B$. Clearly, $u_A(x_A) \geq u_A(x_B)$ and $u_B(x_B) \geq u_B(x_A)$.

  Case 1: $x_A P x_B$. Then there is an open set around x_A of policies that beat x_B, but then $\nabla u_A(x_A) = 0$.

Necessary Conditions (cont.)

Case 2: $x_A I x_B$. Then $x'_A P x_A$, so A can gain by moving slightly toward x_B.

- Note: "nonsatiated" cannot be weakened to "nonsatiated."

Necessary Conditions (cont.)

- **Theorem 2.1:** Assume n odd. Let (x_A, x_B) be a nonaligned equilibrium, so $x_A = x_B = \widehat{x}$. If the candidates' gradients are linearly dependent, then $\widehat{x}$ is a core point.

- Sketch: Suppose $yP\widehat{x}$ for some $y \in X$. If $\nabla u_A \cdot (y - \widehat{x}) > 0$, then y' close enough to $\widehat{x}$ is a profitable deviation for A.

Necessary Conditions (cont.)

If $\nabla u_A \cdot (y - \widehat{x}) = 0$, then we can find z such that $z P \widehat{x}$ and $\nabla u_A \cdot (z - \widehat{x}) > 0$, as before.

- Note: "nonaligned" cannot be weakened to "nonsatiated."

Necessary Conditions (cont.)

- Define

$$\text{cone}^\circ\{p, q\} \;=\; \{\alpha p + \beta q \mid \alpha, \beta > 0\}.$$

- **Theorem 2.2:** Assume n odd. Let (x_A, x_B) be a nonaligned equilibrium, so $x_A = x_B = \widehat{x}$. If the candidates' gradients are linearly independent, then
  - $\widehat{x} = \tilde{x}_k$ for some voter k.
  - for every $p \in \text{cone}^\circ\{\nabla u_A, \nabla u_B\}$,

$$\begin{aligned}&\#\{i \mid \exists \alpha > 0 \text{ s.t. } \nabla u_i = \alpha p\}\\ &\quad = \; \#\{i \mid \exists \alpha < 0 \text{ s.t. } \nabla u_i = \alpha p\}.\end{aligned}$$

Necessary Conditions (cont.)

- A picture of the symmetry condition:

- Sketch: If $\hat{x}$ is no voter's ideal point, then we can find a hyperplane with normal q. Either $> n/2$ voters are on the q side, or $> n/2$ are on the $-q$ side. So some candidate can deviate.

Necessary Conditions (cont.)

So $\widehat{x} = \tilde{x}_k$ for some k. If the symmetry condition is violated, then we do the same thing. But now we find a q' close to $\widehat{x}$ and nudge it in the $\nabla u_i(\widehat{x})$ direction to find a profitable deviation.

Necessary Conditions (cont.)

- **Theorem 3:** Assume n odd. Let (x_A, x_B) be a nonaligned equilibrium, so $x_A = x_B = \widehat{x} = \tilde{x}_k$ for some voter k. For every $p \notin \text{span}\{\nabla u_A, \nabla u_B\}$,

$$\begin{aligned}&\#\{i \mid \exists\alpha > 0 \text{ s.t. } \nabla u_i = \alpha p\} \\ &\quad = \#\{i \mid \exists\alpha < 0 \text{ s.t. } \nabla u_i = \alpha p\}.\end{aligned}$$

- A picture:

Necessary Conditions (cont.)

- **Corollary 1:** Assume n odd. Assume that, for all i, the dimension of

$$\text{span}\{\nabla u_j(\tilde{x}_i) \mid j \in N\}$$

is at least three. Assume that, for all $j, k \neq i$, $\nabla u_j(\tilde{x}_i)$ and $\nabla u_k(\tilde{x}_i)$ are linearly independent. Then there does not exist a nonaligned equilibrium.

- "Nonaligned" cannot be weakened to "nonsatiated" in Theorem 3.

Necessary Conditions (cont.)

- **Theorem 4:** Assume n even. There does not exist a nonaligned equilibrium.

- Sketch: We show that in every nonaligned equilibrium, $x_A = x_B = \widehat{x} = \tilde{x}_k$ for some k. Now delete k, leaving an electorate N' with an odd number of voters, none with ideal point at $\widehat{x}$. By our arguments, there is some $y \in X$, better than $\widehat{x}$ for some candidate, such that $yP'\widehat{x}$. Adding k back to the electorate, we still have $yR\widehat{x}$. So that candidate gains from deviating.

- "Nonaligned" cannot be weakened to "nonsatiated."

Necessary Conditions (cont.)

- Example: Not even locating at the core point is an equilibrium, not even in one dimension.

- Example: Same in two dimensions.

Other Equilibria

- Say x satisfies the *alignment condition* if the candidates' gradients point in the same direction: there exist $\alpha, \beta \geq 0$, at least one nonzero, such that $\alpha \nabla u_A(x) = \beta \nabla u_B(x)$.

- **Proposition 3:** Assume n is odd and, for each x satisfying the alignment condition, there exists $y \in X$ such that yPx and, for some C, $u_C(y) > u_C(x)$. If (x_A, x_B) is an interior equilibrium, then it is nonaligned.

- When n is even, there is the added possibility that the equilibrium is "aligned" with $x_A = \tilde{x}_A$ and $x_B = \tilde{x}_B$.

Other Equilibria (cont.)

- Let Y be the *yolk*, the smallest closed ball intersecting all median hyperplanes.

- If the yolk is "between" $\tilde{x}_A$ and $\tilde{x}_B$, then the condition of Proposition 3 is satisfied.

- In one dimension, this means $\tilde{x}_A$ and $\tilde{x}_B$ are on opposite sides of the median.

Other Equilibria (cont.)

- In multiple dimensions, if the core is nonempty, this means that $\tilde{x}_A$ and $\tilde{x}_B$ do not point in the same direction from x^*.

- The condition is weaker, the smaller is the yolk (the closer the core is to being nonempty).

Robustness

- Consider sequences $\text{Pr}_C^m(\cdot)$ and $w_C^m(\cdot)$ satisfying

  (a) $0 < \text{Pr}_C^m(\cdot) < 1$,

  (b) $w_C^m(\cdot)$ converges uniformly to zero,

  (c) if $x_A P x_B$, then $\text{Pr}_A^m(\cdot)$ converges uniformly to 1 on some neighborhood containing (x_A, x_B) (and similarly for B).

- In game m, candidate C's payoff is:

$$\begin{aligned} U_C^m(x_A, x_B) & \\ = \ & \text{Pr}_A^m(x_A, x_B)u_C(x_A) + \text{Pr}_B^m(x_A, x_B)u_C(x_B) \\ & + \text{Pr}_C^m(x_A, x_B)w_C^m(x_A, x_B). \end{aligned}$$

Robustness (cont.)

- Condition (c) is weak and captures the intuitive notion of "closeness" in the two most common models of probabilistic voting: uncertainty about. . .
  - voter preferences over nonpolicy characteristics of candidates (random, additive utility shocks),
  - voter policy preferences (random voter ideal points).

- **Theorem 5:** Assume n odd. If there is no equilibrium in the model with pure policy motivation and deterministic voting, then there is no equilibrium in model m for m high enough.

Conclusions

- The equilibrium existence problem is nuanced, but fundamental.

- Alternative approaches:

  - mixed strategy equilibria – difficult to compute, perhaps level of rationality is too demanding.

  - repeated elections – if voter discount factors are $> 1/2$, then every path of policies can be supported by s.g.p. equilibrium.

  - probabilistic voting – existence is a problem when close to certainty.

- In lieu of mixing, deeper models of campaigns and elections needed, e.g., electoral accountability, campaign finance.