Generic Expansiveness of the Majority Top Cycle

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Seminal Papers

McKelvey (1976, 1979)

- Fixes preference profile, assumes preference diversity.
- Early paper uses strong assumptions.
- Later paper gives strong necessary conditions on frontier of top cycle.
- General, but complex and a little ambiguous.

Schofield (1978, 1983)

- Early paper gives conditions for local cycles.
- Later paper shows local cycles are dense for a residual set of utility profiles when $d \geq 3$.
- Adding a dimension, $TC = X$ for a residual set of utility profiles.

This Paper

- Simple statement of genericity result in terms of finite-dimensional perturbations.
- Perturbations preserve nice properties, like concavity.
- Proof establishes genericity of McKelvey's preference diversity (clarifies connection to Austen-Smith and Banks')
- Saves a dimension. (I use a "global indifference lemma" due to McKelvey to get an extra restriction.)
- Understandable to undergrads. . .

Social Choice Framework

X	set of alternatives, open and connected in $\Re^d$
N	set of n agents, n odd, $n \geq 3$
P_i	strict preference relation for i
I_i	indifference relation for i
u_i	smooth utility representation for i
s^i	linear preference perturbation for i

Social Choice Framework (cont.)

- Preferences at $s = (s^1, \ldots, s^n)$ are given by

$$U_i(x, s) \;\; = \;\; u_i(x) + s^i \cdot x.$$

- We use $P_i(s)$, $I_i(s)$, $P_i(x, s)$, $I_i(x, s)$ in the predictable way.
- Strict majority preference at s:

$$xP(s)y \quad \text{iff} \quad |\{i \mid xP_i(s)y\}| > \frac{n}{2}$$

- Majority core at s:

$$K(s) \;\; = \;\; \{x \mid \text{there is no } y \text{ s.t. } yP(s)x\}$$

Top Cycle Theorem

- Majority top cycle at s:

$$TC(s) = \left\{ x \mid \begin{array}{l} \text{for all } y\text{, there are } x_1, \dots, x_m \text{ s.t.} \\ xP(s)x_1P(s)\cdots x_{m-1}P(s)x_m = y \end{array} \right\}$$

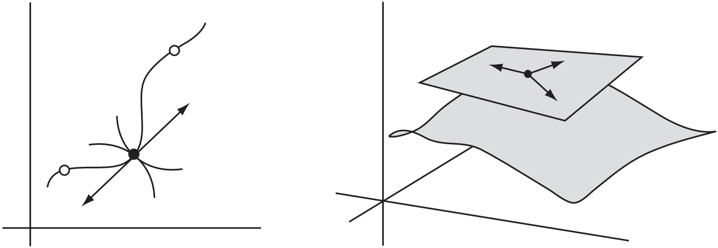
- Global Cycle Theorem: Assume $d \geq 3$. Then for almost all s, the top cycle at s equals X.

Pareto Manifolds

- Pareto Manifold Lemma 1: For almost all s and for all coalitions C with $|C| \leq d+1$, the set

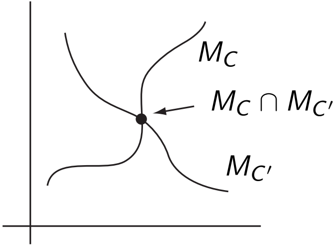
$$M_C(s) = \left\{ x \mid \text{rank}\{Du_i(x) + s^i \mid i \in C\} = |C| = 1 \right\}$$

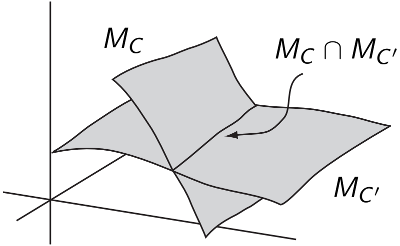
is a $(|C|-1)$-dimensional manifold.



Pareto Manifolds (cont.)

- Pareto Manifold Lemma 2: For almost all s and all non-nested C, C' with $\max\{|C|, |C'|\} \leq d+1$, the set $M_C(s) \cap M_{C'}(s)$ is a manifold of dimension $|C| + |C'| - d - 2$.
- For example, if $|C| = |C'| = 3$, then the intersection has dimension $3 + 3 - d - 2 = 4 - d \leq 1$.





Preference Diversity

- Weak preference diversity at s: For all x and all distinct i, j such that $Du_i(x) + s^i \neq 0$, the set $I_i(x, s) \cap I_j(x, s)$ has empty interior in $I_i(x, s)$.
- The frontier of a set Y is

$$\text{fr}\, Y \quad = \quad \text{clos}(\text{int}\, Y) \cap \text{clos}(\text{int}\, \overline{Y}).$$

- Strong preference diversity at s: For all Y, all $x \in \text{fr}\, Y$, and all distinct i, j, the set $I_i(x, s) \cap I_j(x, s)$ has empty interior in $\text{fr}\, Y$.

generic s $\Rightarrow$ weak pref. diversity $\Rightarrow$ strong pref. diversity

Constrained Symmetry

- Let $T(x, s)$ be the alternatives that directly or indirectly dominate x, i.e.,

$$T(x, s) = \left\{ y \mid \begin{array}{c} \text{there are } x_1, \ldots, x_m \text{ such that} \\ yP(s)x_1P(s)\cdots x_{m-1}P(s)x_m = x \end{array} \right\}.$$

- Global Indifference Lemma: For almost all s and for all x, there is some agent k such that u_k is constant on $\operatorname{fr} T(x, s)$, i.e., for some y, we have $\operatorname{fr} T(x, s) \subseteq I_k(y, s)$.

Constrained Symmetry (cont.)

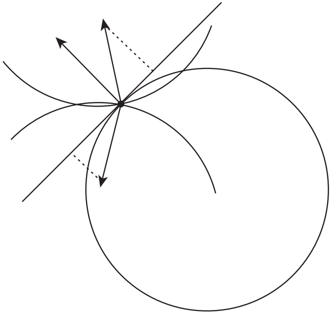
- Constrained Symmetry Theorem: Assume $d \geq 3$. For generic s, let k be indifferent on $\operatorname{fr} T(x, s)$. Then constrained symmetry holds at all $y \in \operatorname{fr} T(x, s)$, i.e., there is a partition $\{J_1, \ldots, J_m\}$ of $N \setminus \{k\}$ such that for all $J_\ell = \{i, j\}$, we have

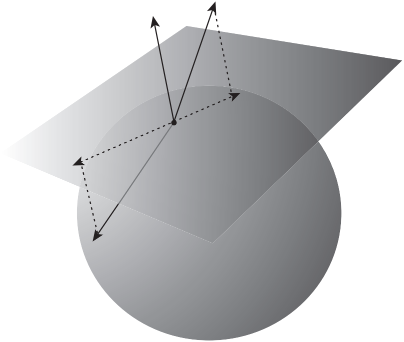
$$Du_k(y) + s^k \in \operatorname{span}\{Du_i(y) + s^i, Du_j(y) + s^j\}.$$

- An implication is that

$$\operatorname{fr} T(x, s) \quad \subseteq \quad \bigcup_{\substack{C:|C|=1,2,3 \\ k \in C}} M_C(s).$$

Constrained Symmetry (cont.)



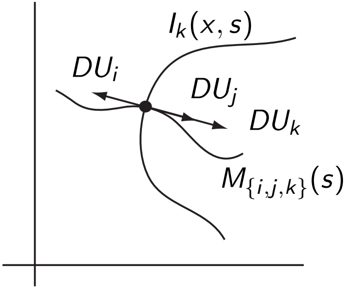


Idea of Proof

- It suffices to show that $X \subseteq \operatorname{clos} T(x, s)$ for all x, so suppose $X \setminus \operatorname{clos} T(\tilde{x}, s)$ is nonempty.
- We can show that there are agents i and j and an open set Z such that:
  (i) $Z \cap \operatorname{fr} T(\tilde{x}, s) \subseteq M_{\{i,j,k\}}(s)$
  (ii) $Z \cap \operatorname{fr} T(\tilde{x}, s) = Z \cap I_k(x, s)$.
- But $M_{\{i,j,k\}}(s)$ is a lower dimensional set that intersects $I_k(x, s)$ transversally...

Idea of Proof (cont.)

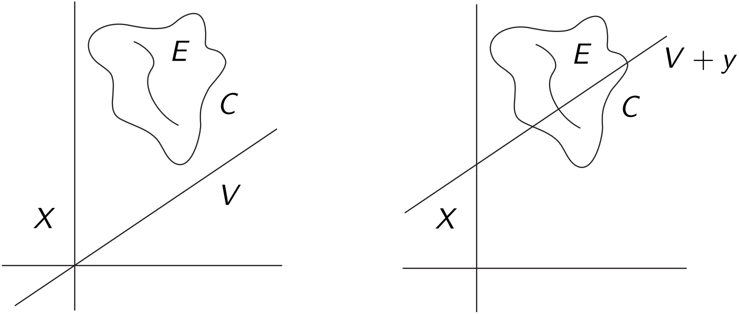
- A side view:



- So $M_{\{i,j,k\}}(s) \cap I_k(x, s)$ has dimension less than or equal to $d - 2$, and we cannot have $Z \cap I_k(x, s) \subseteq M_{\{i,j,k\}}(s)$, a contradiction.

Finite Shyness

- Let X be a completely metrizable tvs and $C \subseteq X$ a convex subset. Then a Borel set $E \subseteq C$ is finitely shy in C if there is a finite-dimensional subspace $V \subseteq C$ such that $\lambda_{V+x}(C) > 0$ for some $x \in X$ and $\lambda_{V+y}(E) > 0$ for every $y \in X$.



Finite Shyness (cont.)

- Map Anderson and Zame (2001) to our framework:

X	n-fold product space of smooth functions on $\Re^d$
E	space of $(u_1, \ldots, u_n)$ with top cycle not equal to X
V	n-fold product space of linear functions on $\Re^d$

- Global Cycle Corollary 1: The space of utility profiles for which the top cycle is not equal to X is finitely shy in the space of smooth profiles.
- Global Cycle Corollary 2: The space of utility profiles for which the top cycle is not equal to X is finitely shy in the space of smooth, concave profiles.

Extras

- The results just need an extra dimension if X is a differentiable manifold with boundary.
- Suppose the set of alternatives is any set $X' \subseteq \Re^d$. Then the theorem holds for any open, connected subset $X \subseteq X'$, giving us arbitrarily tight cycles around any pre-specfied path through X'.
- The n even case presents different challenges. Expansiveness of the top cycle is generic as long as $d \geq 4$ and $n \geq 6$.