Assignment 2 (Due Thursday, October 12)

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1. The Pareto distribution, with parameters $\alpha > 0$ and $\beta > 0$, has pdf,

$$f_X(x;\alpha,\beta) = \frac{\beta\alpha^\beta}{x^{\beta+1}} 1\{\alpha \le x < \infty\}$$

(a) Verify that $f(x)$ is a pdf.

(b) Show that $E[X] = \dfrac{\alpha\beta}{\beta - 1}$ when $\beta > 1$, and that the mean does not exist if $\beta \le 1$.

(c) Show that $Var(X) = \dfrac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$ when $\beta > 2$, and that the variance does not exist when $\beta \le 2$.

2. A generalization of the Cauchy distribution has pdf,

$$f_X(x;\mu,\sigma) = \frac{1}{\sigma\pi\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)}$$

The mean and variance of the Cauchy distribution do not exist, so μ and σ are not the mean and variance, but they do have an important meaning. Show that if X is a Cauchy random variable with parameters μ and σ then,

(a) μ is the median of the distribution, that is, $P(X \ge \mu) = P(X \le \mu) = \frac{1}{2}$

(b) $\mu - \sigma$ and $\mu + \sigma$ are the quartiles of the distribution of X, that is,

$P(X \geq \mu + \sigma) = P(X \leq \mu - \sigma) = \frac{1}{4}$ (Hint: use the transformation, $y = \dfrac{x-\mu}{\sigma}$ and

use $\int \frac{dx}{1+x^2} = \tan^{-1}(x)$, $\tan^{-1}(-\infty) = -\frac{\pi}{2}$, and $\tan^{-1}(-1) = \frac{\pi}{4}$).

3. A random point (X, Y) is distributed uniformly on the square with vertices (1,1), (1,-1), (-1,1), and (-1,-1). That is, the joint pdf is $f(x, y) = \frac{1}{4}$ on the square. Determine the probabilities of the following events.

(a) $X^2 + Y^2 < 1$

(b) $2X - Y > 0$

(c) $|X + Y| < 2$

4. A pdf is defined by,

$$f(x, y) = \begin{cases} c(x + 2y), & 0 < y < 1, 0 < x < 2 \\ 0, & otherwise \end{cases}$$

(a) Find the value of c

(b) Find the marginal distribution of X

(c) Find the joint cdf of X and Y

(d) Find the pdf of the random variable $Z = 9/(X+1)^2$

5. Let U be the number of trials needed to get the first head and V be the number of trials needed to get two heads in repeated tosses of a fair coin. Are U and V independent random variables? (Hint: show that $P(V = k \mid U = j) \neq P(V = k)$).

6. Let X and Y be independent N(0,1) random variables, and define a new random variable Z by,

$$Z = \begin{cases} X, & XY \geq 0 \\ -X, & XY < 0 \end{cases}$$

(a) Show that Z has a normal distribution.

(b) Show that the joint distribution of Z and Y is not bivariate normal. (Hint: show that Z and Y always have the same sign).