Political Science 404
Problem Set 4 Solution

1 Suppose that $X_n \sim N(\mu, \sigma^2)$. Then $\frac{\bar{X}^{N+1}-\mu}{s/\sqrt{N+1}} \sim t_N$. By the CLT, we have that $\frac{\bar{X}^{N+1}-\mu}{s/\sqrt{N+1}} \to N(0,1)$, thus $t_n \to N(0,1)$.

2 Recall that if $X^N \sim \text{Binomial}(N, \frac{\lambda}{N})$, then X^N has characteristic function $\varphi_{X^N}(t) = (1 - \frac{\lambda}{N} + \frac{\lambda}{N}e^{it})^N$. Taking logs of both sides gives,

$$\begin{aligned} \log\varphi_{X^N}(t) &\to \frac{-\lambda(1-e^{it})}{1-\lambda N^{-1}(1-e^{it})} \to -\lambda(1-e^{it}) \\ \Rightarrow \varphi_{X^N}(t) &\to e^{-\lambda(1-e^{it})} = e^{\lambda(e^{it}-1)} \end{aligned}$$

Since this is the characteristic function of a Poisson random variable, the result holds.

The CLT fails to hold because the probability of success is changing with N in this case.

3 We have

$$\varphi_{\bar{X}}(t) = \varphi_{\sum_{n=1}^{N} X_n}(\frac{1}{N}t) = \varphi_{X_1}(\frac{1}{N}t)^N = e^{-N\gamma|\frac{1}{N}t|} = e^{-\gamma|t|} = \varphi_{X_1}(t)$$

The above uses the fact

$$\varphi_{cX}(t) = E[e^{citx}] = \varphi_X(ct).$$

The LLN fails to hold because the expectation of a Cauchy random variable fails to exist. The CLT fails to hold because the Cauchy random variable does not have finite variance.

4 First we know s^2 is a consistent estimator of σ^2 (see the proof on page 233 (ex. 5.5.3) in the textbook). Next, let $g(X) = \sqrt{X}$. The continuous mapping theorem tells us if $s^2 \to \sigma^2$ we have $s \to \sigma$ (ex. 5.5.5).

5 Here both methods give the same answer.

$$P(\bar{X} < 5.2) = P(\frac{\bar{X}-6.5}{1.1/\sqrt{31}} < \frac{5.2-6.5}{1.1/\sqrt{31}}) = \Phi(-6.58) = 2.352e-11$$

6 The exact answer gives $\frac{\bar{X}-\mu}{s/\sqrt{N}} \sim t_{16}$, so

$$P(-2 < \frac{\bar{X}-\mu}{s/\sqrt{N}} < 2) = .937$$

The approximation answer gives $\frac{\bar{X}-\mu}{s/\sqrt{N}} \sim N(0,1)$, so

$$P(-2 < \frac{\bar{X}-\mu}{s/\sqrt{N}} < 2) = \Phi(2) - \Phi(-2) = .955$$

7 The exact answers gives

$$P(54 \leq 200p \leq 64) = \sum_{x=54}^{64} \begin{pmatrix} 200 \\ x \end{pmatrix} 0.3^x 0.7^{200-x} = .6$$

The approximation answer gives

$$\begin{aligned} P(.27 \leq p \leq .32) &= P(\frac{.27-.3}{\sqrt{.3*.7/200}} \leq \frac{p-.3}{\sqrt{.3*.7/200}} \leq \frac{.32-.3}{\sqrt{.3*.7/200}}) \\ &= \Phi(.617) - \Phi(-.926) = .554 \end{aligned}$$