Assignment 5 (Due Thursday, November 30)

---

1. Suppose that $X_n \sim N(\mu, \sigma^2)$ and that we find that $\bar{X} = 13$ and $s = 2.4$ in a sample of size 17. Test the null hypothesis that $\mu = 15$ at the 1% level. Provide an exact answer as well as an approximation based on the Central Limit Theorem.

2. Suppose that $X_n \sim N(\mu, \sigma^2)$ and we find that $s^2 = 194$ in a sample of size 23. Form a 98% confidence interval for σ^2.

3. Suppose that $X_n \sim N(\mu_Y, \sigma_Y{}^2)$ and $Y_n \sim N(\mu_Y, \sigma_Y{}^2)$. We find that $s_X{}^2 = 12$ and $s_Y{}^2 = 20$ in samples of size $N_X = 4$ and $N_Y = 29$. Test the null hypothesis that $\sigma_X{}^2 = \frac{1}{2}\sigma_Y{}^2$ at the 10% level.

4. Suppose that X_n has the geometric distribution for $n = 1, 2, \ldots, N$, where $P(X_n = k) = (1 - \pi_0)^{k-1}\pi_0$ for $k = 1, 2, \ldots$.

(a) Compute the likelihood and log-likelihood functions for model.

(b) Find an expression for the maximum likelihood estimator.

(c) Find a method of moments estimator using the moment $P(X_n \leq 2)$ (Hint: you will have to use the Quadratic formula).

5. Suppose that X_n has the log-normal distribution with

$$f(x;\mu,\sigma)=\frac{1}{\sigma x\sqrt{2\pi}}e^{-\frac{1}{2}(\log x-\mu)^2/\sigma^2} \text{ for } x\geq 0$$

(a) Find the maximum likelihood estimator of (μ,σ)

(b) Find the method of moments estimator using the mean and variance of the log-normal (Hint $E[X]=e^{\mu+\sigma^2/2}$ and $Var(X)=(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$). Are the mle and method of moments estimators the same?

6. The Poisson regression model is used to model a dependent variable that takes on the positive integers as values. The model assumes that y_n has a Poisson distribution with mean $e^{\beta_0^T x_n}$ (recall that the Poisson distribution has pmf $P(y_n=k)=\dfrac{e^{-\lambda}\lambda^k}{k!}$).

(a) What are the likelihood and log-likelihood functions?

(b) Derive the first-order conditions for the maximum likelihood estimator. Argue that the maximum likelihood estimator does not have a closed for solution (and hence must be computed using a computer).

(c) Consider the semi-parametric model where y_n takes on the values 0,1,2,…, and $E[y_n]=e^{\beta_0 x_n}$. Show, using the first-order condition computed in part (b), that the maximum likelihood estimator identifies β_0 even when y_n does not have the Poisson distribution with mean $e^{\beta_0 x_n}$, as long as $E[y_n]=e^{\beta_0 x_n}$ holds.

7. Consider the model, $y_n=\beta_0^T x_n+\varepsilon_n$ where (i) $E[\varepsilon_n \mid x_n]=0$, (ii) (x_n,ε_n) are independent and identically distributed across n, and (iii) $E[x_n x_n^T]$ is an invertible matrix.

(i) Show that the OLS estimator, $\hat{\beta} = \left[\frac{1}{N} \sum_{n=1}^{N} x_n x_n^{\,T} \right]^{-1} \left[\frac{1}{N} \sum_{n=1}^{N} x_n y_n \right]$, is asymptotically normal, in the sense that $\sqrt{N}(\hat{\beta} - \beta_0) \xrightarrow{dist.} N(0, Q_{xx}^{\;-1} V_{x\varepsilon} Q_{xx}^{\;-1})$ where $Q_{xx} = E[x_n x_n^{\,T}]$ and $V_{x\varepsilon} = Var(x_n \varepsilon_n)$. (Hint: first show that $\sqrt{N}(\hat{\beta} - \beta_0) = \left[\frac{1}{N} \sum_{n=1}^{N} x_n x_n^{\,T} \right]^{-1} \left[\frac{1}{\sqrt{N}} \sum_{n=1}^{N} x_n \varepsilon_n \right]$, as we did in class. Then apply a law of large numbers to $\frac{1}{N} \sum_{n=1}^{N} x_n x_n^{\,T}$ and a central limit theorem to $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} x_n \varepsilon_n$, and apply Slutsky's theorem).

(ii) Is OLS a parametric, semi-parametric, and non-parametric estimator?