Political Science 404
Problem Set 5 Solution

1 We have $H_0 : \mu = 15$ and $H_A : \mu \neq 15$, where $\alpha = .01$.

The exact answer gives

$$t = \frac{13 - 15}{2.4/\sqrt{17}} = -3.44$$

and the critical value for t_{16} at $\alpha = .01$ is 2.92. Since $|t| > t_{16,\alpha/2}$, we reject the null.

The approximation gives

$$Z = \frac{13 - 15}{2.4/\sqrt{17}} = -3.44$$

and the critical value for Z at $\alpha = .01$ is 2.57. Since $|Z| > 2.57$, we reject the null.

2 We know the confidence interval for σ^2 is given by

$$[\frac{(N-1)s^2}{\chi^2_{N-1,1-\alpha/2}}, \frac{(N-1)s^2}{\chi^2_{N-1,\alpha/2}}].$$

In this problem $s^2 = 194, N = 23,$ and $\alpha = .02$, so $\chi^2_{N-1,\alpha/2} = 9.54$ and $\chi^2_{N-1,1-\alpha/2} = 40.29$. Therefore, the confidence interval is $[105.9,\ 447.4]$.

3 We have $H_0 : \sigma^2_X = \frac{1}{2}\sigma^2_Y$ and $H_A : \sigma^2_X \neq \frac{1}{2}\sigma^2_Y$, where $\alpha = .1$.

Use

$$\frac{s^2_x/\sigma^2_x}{s^2_y/\sigma^2_y} \sim F_{N_x-1,\ N_y-1}$$

we get $\frac{12\times 2}{20} = 1.2$, $F_{3,28} = 2.95$ at $\alpha = .95$ and $F_{3,28} = .12$ at $\alpha = .05$. Since $.12 < 1.2 < 2.95$, we accept the null.

4a The likelihood function is

$$L = \prod_{n=1}^{N} (1-\pi)^{X_n - 1}\pi$$

The log-likelihood function is

$$l = \sum_{n=1}^{N} [(X_n - 1)\log(1-\pi) + \log\pi]$$

4b

$$\frac{d}{d\pi} l = \sum_{n=1}^{N} [\frac{-(X_n - 1)}{(1-\hat{\pi})} + \frac{1}{\hat{\pi}}] = 0$$

$$\Rightarrow \hat{\pi} = \frac{1}{\bar{X}}$$

4c The theoretical moment is

$$P(X_n \leq 2) = (1-\pi)^{1-1}\pi + (1-\pi)^{2-1}\pi = -\pi^2 + 2\pi = 1 - (1-\pi)^2,$$

and the empirical moment is $\frac{1}{N}\sum_{n=1}^{N} 1\{X_n \leq 2\}$.

Let $\frac{1}{N}\sum_{n=1}^{N} 1\{X_n \leq 2\} = 1 - (1-\hat{\pi})^2$, we get $\hat{\pi} = 1 - \sqrt{1 - \frac{1}{N}\sum_{n=1}^{N} 1\{X_n \leq 2\}}$.

5a The log-likelihood function is

$$l = \sum_{n=1}^{N}[-\log\sigma - \log X_n\sqrt{2\pi} - \frac{1}{2}(\log X_n - \mu)^2/\sigma^2].$$

$$\text{so} \quad \frac{d}{d\mu}l = \sum_{n=1}^{N}[(\log X_n - \hat{\mu})/\hat{\sigma}^2] = 0$$

$$\Rightarrow \quad \hat{\mu} = \frac{1}{N}\sum_{n=1}^{N}\log X_n$$

$$\text{and} \quad \frac{d}{d\sigma}l = \sum_{n=1}^{N}[-\frac{1}{\hat{\sigma}} + (\log X_n - \hat{\mu})^2/\hat{\sigma}^3] = 0$$

$$\Rightarrow \quad \hat{\sigma} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(\log X_n - \hat{\mu})^2}$$

5b We have

$$\bar{X} = e^{\hat{\mu}+\hat{\sigma}^2/2} \text{ and } \frac{N-1}{N}s^2 = (e^{\hat{\sigma}^2} - 1)e^{2\hat{\mu}+\hat{\sigma}^2},$$

so we can solve these two equations and get

$$\hat{\mu} = \log\bar{X} - \frac{1}{2}\log(1 + \frac{\frac{N-1}{N}s^2}{\bar{X}^2}) \text{ and } \bar{\sigma}^2 = \log(1 + \frac{\frac{N-1}{N}s^2}{\bar{X}^2}).$$

6a The likelihood function is

$$L = \prod_{n=1}^{N}\frac{e^{-e^{\beta^T x_n}}(e^{\beta^T x_n})^{y_n}}{y_n!}$$

and the log-likelihood function is

$$l = \sum_{n=1}^{N}[-e^{\beta^T x_n} + y_n(\beta^T x_n) - \log y_n!]$$

6b

$$\frac{d}{d\beta}l \quad = \quad \sum_{n=1}^{N}x_n[-e^{\hat{\beta}^T x_n} + y_n] = 0$$

6c Since $E[y_n] = e^{\beta_0^T x_n}$, we have

$$E[x_n[-e^{\hat{\beta}^T x_n} + y_n]] = E[x_n E_x[-e^{\hat{\beta}^T x_n} + y_n|x_n]] = E[x_n \times 0] = 0.$$

Also, we want to show $E[x_n[e^{\beta_0^T x_n} - e^{\beta^T x_n}]] \neq 0$ for $\beta \neq \beta_0$. Let

$$\int x[e^{\beta^T x} - e^{\beta_0^T x}] f_x(x) dx = 0.$$

Using Taylor expansion

$$e^{\beta^T x} = e^{\beta_0^T x} + x^T e^{\bar{\beta}(x)^T x}(\beta - \beta_0),$$

the above becomes

$$\int x x^T e^{\bar{\beta}(x)^T x} f_x(x) dx (\beta - \beta_0) = 0.$$

Since $\int x x^T e^{\bar{\beta}(x)^T x} f_x(x) dx \neq 0$, we must have $(\beta - \beta_0) = 0$, which implies $\beta = \beta_0$.

7a First, since

$$\begin{aligned}
\hat{\beta} &= [\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T]^{-1}[\frac{1}{N}\sum_{n=1}^{N} x_n y_n] \\
&= [\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T]^{-1}[\frac{1}{N}\sum_{n=1}^{N} x_n(\beta_0^T x_n + \epsilon_n)] \\
&= [\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T]^{-1}[\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T \beta_0] \\
&\quad +[\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T]^{-1}[\frac{1}{N}\sum_{n=1}^{N} x_n \epsilon_n] \\
&= \beta_0 + [\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T]^{-1}[\frac{1}{N}\sum_{n=1}^{N} x_n \epsilon_n],
\end{aligned}$$

we get

$$\sqrt{N}(\hat{\beta} - \beta_0) = [\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T]^{-1}[\frac{1}{\sqrt{N}}\sum_{n=1}^{N} x_n \epsilon_n].$$

Next,

$$E[\frac{1}{\sqrt{N}}\sum_{n=1}^{N} x_n \epsilon_n] = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} E[x_n \epsilon_n] = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} E[x_n E_x[\epsilon_n | x_n]] = 0$$

because $E[\epsilon_n | x_n] = 0$, and

$$Var[\frac{1}{\sqrt{N}}\sum_{n=1}^{N} x_n \epsilon_n] = \frac{1}{N}\sum_{n=1}^{N} Var[x_n \epsilon_n] = V_{x\epsilon}.$$

Also, by the law of large numbers, $\frac{1}{N}\sum_{n=1}^{N} x_n x_n^T \to E[x_n x_n^T] = Q_{xx}$.

Put everything together, we have the following based on a central limit theorem

$$\begin{aligned} & \frac{[\frac{1}{\sqrt{N}}\sum_{n=1}^{N} x_n\epsilon_n - 0]}{V_{x\epsilon}} \to N(0,1) \\ \text{or} \quad & [\frac{1}{\sqrt{N}}\sum_{n=1}^{N} x_n\epsilon_n - 0] \to N(0, V_{x\epsilon}) \\ \Rightarrow \quad & \sqrt{N}(\hat{\beta} - \beta_0) \to N(0, Q_{xx}^{-1} V_{x\epsilon} Q_{xx}^{-1}), \end{aligned}$$

where in the last step Slutsky's theorem is applied.

7b OLS is a semi-parametric estimator.